

EXPLICIT VERSIONS OF THE LOCAL DUALITY THEOREM IN \mathbb{C}^n

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ABSTRACT. We consider versions of the local duality theorem in \mathbb{C}^n . We show that there exist canonical pairings in these versions of the duality theorem which can be expressed explicitly in terms of residues of Grothendieck, or in terms of residue currents of Coleff-Herrera and Andersson-Wulcan, and we give several different proofs of non-degeneracy of the pairings. One of the proofs of non-degeneracy uses the theory of linkage, and conversely, we can use the non-degeneracy to obtain results about linkage for modules. We also discuss a variant of such pairings based on residues considered by Passare, Lejeune-Jalabert and Lundqvist.

1. INTRODUCTION

Let $\mathcal{O} = \mathcal{O}_{\mathbb{C}^n, 0}$ be the local ring of germs of holomorphic functions at $0 \in \mathbb{C}^n$. We will in this article mainly consider the variant of the local duality theorem of Grothendieck, as presented by Hartshorne in [H, Theorem 6.3], see Theorem 2.1 below. We will also later discuss a variant of this expressed as integrals, generalizing the variant of the local duality theorem for Artinian complete intersection ideals as presented in [GH, p. 693].

Let G be a finitely generated \mathcal{O} -module. We let

$$G_{(p)} := \{g \in G \mid \text{codim}(\text{supp } g) \geq p\}.$$

The torsion elements of G are precisely the elements in $G_{(1)}$. For any G , there is a natural pairing $G \times \text{Hom}(G, \mathcal{O}) \rightarrow \mathcal{O}$, given by $(g, \varphi) \mapsto \varphi(g)$. Since \mathcal{O} is torsion-free, elements of $G_{(1)}$ are mapped to 0, so the pairing descends to a pairing

$$(1.1) \quad G/G_{(1)} \times \text{Hom}(G, \mathcal{O}) \rightarrow \mathcal{O},$$

and it is classical that this pairing is non-degenerate, see for example [GR, 3.3.3, p. 69].

If $\text{codim } G \geq 1$, then $G/G_{(1)} = 0$, so the pairing (1.1) is uninteresting, but we consider here a variant of it. For a germ of a subvariety $Z \subseteq (\mathbb{C}^n, 0)$, we let $H_Z^p(\mathcal{O})$ denotes the p -th local cohomology module of \mathcal{O} with support in Z , see Section 2.1.

Theorem 1.1. *Let G be a finitely generated \mathcal{O} -module of codimension $\geq p$, and let $Z \supseteq \text{supp } G$, where Z has pure codimension p . There exists a canonical explicitly given pairing*

$$(1.2) \quad G \times \text{Ext}^p(G, \mathcal{O}) \rightarrow H_Z^p(\mathcal{O}),$$

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which is functorial in G , and which descends to a non-degenerate pairing

$$(1.3) \quad G/G_{(p+1)} \times \operatorname{Ext}^p(G, \mathcal{O}) \rightarrow H_Z^p(\mathcal{O}).$$

The pairing (1.2) can be defined by either (2.9) or (6.9).

When we say that the pairing is functorial in G , we mean that the following diagram commutes for any finitely generated \mathcal{O} -module F of codimension $\geq p$ and morphism $\alpha : F \rightarrow G$, where $Z \subseteq (\mathbb{C}^n, 0)$ has pure codimension p and $Z \supseteq (\operatorname{supp} F) \cup (\operatorname{supp} G)$.

$$(1.4) \quad \begin{array}{ccc} G \times \operatorname{Ext}^p(G, \mathcal{O}) & \longrightarrow & H_Z^p(\mathcal{O}) \\ \alpha \uparrow & \alpha^* \downarrow & \parallel \\ F \times \operatorname{Ext}^p(F, \mathcal{O}) & \longrightarrow & H_Z^p(\mathcal{O}) \end{array}$$

Note that by Lemma 4.3, the choice of Z is not really important.

In the case when $p = n$, Theorem 1.1 becomes a special case of the local duality theorem of Grothendieck, [H, Theorem 6.3], see Remark 2.2. For general p , that there exist a pairing (1.2) is a straightforward generalization of construction of the pairing in [H]. However, the formulation of the fact that this descends to a non-degenerate pairing (1.3) is to our knowledge new, when $p < n$. In addition, even in the already known case, when $p = n$, our proof is quite different from the proof in [H, Theorem 6.3].

This theorem is also very close to the duality theorem of Andersson in [A2, Theorem 1.2], which deals with the case when G has pure codimension p , although there, functoriality is not proven. In [A2], this pairing is defined with the help of certain residue currents constructed by Andersson and Wulcan, [AW1]. It is also proven that this pairing coincides with the pairing from [H] (without explicitly referring to it). We give a direct proof of functoriality for this pairing using a comparison formula for residue currents from [L3].

We consider now the special case when $G = \mathcal{O}/I$, where I is a complete intersection ideal of codimension p , i.e., I can be defined by p functions $I = J(f_1, \dots, f_p)$. In this case, one can define the pairing (1.2) with the help of residue currents of Coleff and Herrera, [CH], and the non-degeneracy of the pairing then becomes the duality theorem for Coleff-Herrera products, as proven independently by Passare, [P], and Dickenstein-Sessa, [DS1], see Section 3 below. For certain morphisms between such complete intersection modules, one obtains functoriality from the transformation law for Coleff-Herrera products, [DS1] or [DS2], see Example 3.4 and (3.11).

We give three different proofs of the non-degeneracy of the pairing in Theorem 1.1. One proof which is heavily based on homological algebra in Section 5, another based on the theory of residue currents, similar to the proof in [A2], in Section 6, and a proof using the theory of linkage in Section 4, although this last proof only gives non-degeneracy in the first argument when G is of the form \mathcal{O}/J for some ideal $J \subseteq \mathcal{O}$.

We now return to the pairing (1.1). That it is non-degenerate in the first argument can be reformulated as that the morphism

$$(1.5) \quad G/G_{(1)} \rightarrow \operatorname{Hom}(\operatorname{Hom}(G, \mathcal{O}), \mathcal{O})$$

given by $g \mapsto (\varphi \mapsto \varphi(g))$ is injective. It is also well-understood when (1.5) is an isomorphism, which is the case if and only if $G/G_{(1)}$ is S_2 , see for example [ST, Corollary 1.21]. By S_2 , we refer to the Serre S_k -conditions, see Section 5.1. Note also that the other morphism induced by (1.1) is the morphism $\text{Hom}(G, \mathcal{O}) \rightarrow \text{Hom}(G, \mathcal{O})$ given by $\varphi \mapsto (g \mapsto \varphi(g))$, which is clearly always an isomorphism. Our next result is that this holds also for the pairing (1.3).

Theorem 1.2. *Let G and Z be as in Theorem 1.1. The injection*

$$(1.6) \quad G/G_{(p+1)} \rightarrow \text{Hom}(\text{Ext}^p(G, \mathcal{O}), H_Z^p(\mathcal{O}))$$

induced by the non-degenerate pairing (1.3) is an isomorphism if and only if $G/G_{(p+1)}$ is S_2 , and the injection

$$(1.7) \quad \text{Ext}^p(G, \mathcal{O}) \rightarrow \text{Hom}(G/G_{(p+1)}, H_Z^p(\mathcal{O}))$$

induced by the non-degenerate pairing (1.3) is always an isomorphism.

If $\text{codim } G = n$, then $G = G/G_{(n+1)}$ is Cohen-Macaulay, and hence also S_2 , so both (1.6) and (1.7) are isomorphisms, which follows from [H]. In case when G has pure codimension $p \leq n$, then (1.7) follows from [A2, Theorem 1.5]. The remaining cases of Theorem 1.2 when $\text{codim } G < n$ are to our knowledge new.

We finally also consider the case when G does not necessarily have codimension $\geq p$. We let $G^{(p)} := G_{(p)}/G_{(p+1)}$, which is either 0 or has pure codimension p .

Theorem 1.3. *Let G be a finitely generated \mathcal{O} -module, and let $Z \supseteq \text{supp } G$, where Z has pure codimension p . There exists a canonical explicitly given pairing*

$$(1.8) \quad G_{(p)} \times \text{Ext}^p(G, \mathcal{O}) \rightarrow H_Z^p(\mathcal{O}),$$

which is functorial in G , and which descends to a non-degenerate pairing

$$(1.9) \quad G^{(p)} \times \text{Ext}^p(G, \mathcal{O})^{(p)} \rightarrow H_Z^p(\mathcal{O}).$$

The injection

$$(1.10) \quad G^{(p)} \rightarrow \text{Hom}(\text{Ext}^p(G, \mathcal{O})^{(p)}, H_Z^p(\mathcal{O}))$$

induced by the non-degenerate pairing (1.9) is an isomorphism if and only if $G^{(p)}$ is S_2 , and the injection

$$(1.11) \quad \text{Ext}^p(G, \mathcal{O})^{(p)} \rightarrow \text{Hom}(G^{(p)}, H_Z^p(\mathcal{O}))$$

induced by the non-degenerate pairing (1.9) is an isomorphism if and only if $\text{Ext}^p(G, \mathcal{O})^{(p)}$ is S_2 . If G has codimension $\geq p$, then (1.2) and (1.8) coincide. The pairing (1.8) can be defined by either (2.9) or (6.9).

We note two things which are implied from the formulation of these theorems. First of all, in (1.3), we have $\text{Ext}^p(G, \mathcal{O})$ in the second argument, while in (1.9), we have $\text{Ext}^p(G, \mathcal{O})^{(p)}$ in the second argument. Both these pairings are non-degenerate, and in the case when G has codimension $\geq p$, these should coincide, i.e., $\text{Ext}^p(G, \mathcal{O})$ must have pure codimension p (or be

0). This is indeed well-known, see for example [EHV, Theorem 1.1]. Secondly, if G is still assumed to be of codimension $\geq p$, then (1.7) and (1.11) coincide, and the first morphism is always an isomorphism, while the second is an isomorphism if and only if $\text{Ext}^p(G, \mathcal{O})^{(p)}$ is S_2 . Thus, $\text{Ext}^p(G, \mathcal{O})^{(p)}$ is always S_2 when G has codimension $\geq p$. This fact, we have not been able to find explicitly stated in the literature, although it can be proven using a result of Björk, see Remark 5.14.

We now consider a consequence of Theorem 1.3 and Theorem 1.2, which can be seen as another generalization of (1.5). We consider an arbitrary finitely generated \mathcal{O} -module G . Then, we have the injection (1.10), which is an isomorphism if and only if $G^{(p)}$ is S_2 . In addition, $\text{Ext}^p(G, \mathcal{O})$ has codimension $\geq p$, see Proposition 2.3, so using (1.7), we obtain the following corollary.

Corollary 1.4. *Let G be a finitely generated \mathcal{O} -module. Then there is a natural injective map*

$$(1.12) \quad G^{(p)} \rightarrow \text{Ext}^p(\text{Ext}^p(G, \mathcal{O}), \mathcal{O}),$$

which is surjective if and only if $G^{(p)}$ is S_2 .

This result is similar to the fundamental theorem of Roos, [R], for which the results and proof has been elaborated by Björk, in for example [B1, Chapter 2]. Part of the theorem of Roos states that the map (1.12) is injective, and that it is surjective outside a set of codimension $\geq p + 2$. The surjectiveness outside a set of codimension $\geq p + 2$ is in fact also enough to prove that it is an isomorphism if and only if it is S_2 . Note that by Remark 5.14, $\text{Ext}^p(\text{Ext}^p(G, \mathcal{O}), \mathcal{O})$ is S_2 , and by Corollary 5.4, (1.12) is then surjective if and only if $G^{(p)}$ is S_2 .

In [A2], it is explained that a version from [A2] of Theorem 1.1 when $G = G^{(p)}$ has pure codimension p , follows from Corollary 1.4, and also that the form of Corollary 1.4 from [R] can be used to prove the version of Theorem 1.1 from [A2].

We will now relate our results with a result from the theory of linkage. We first recall that if I and J are two ideals in a ring R , then $I : J$ is the ideal

$$I : J = \{r \in R \mid rJ \subseteq I\}.$$

Let $J \subseteq \mathcal{O}$ be an ideal of codimension p . The equidimensional hull $J_{[p]}$ of J is the intersection of all primary components of J of p , cf., Remark 4.7. The following result can be found in (the proof of) [V, Proposition 3.41]. If $f = (f_1, \dots, f_p)$ is a regular sequence in J , generating the ideal I , then

$$(1.13) \quad J_{[p]} = I : (I : J).$$

This is a generalization of the case when \mathcal{O}/J is Cohen-Macaulay from the fundamental article [PS] by Peskine and Szpiro.

If G is a finitely generated \mathcal{O} -module, then G is isomorphic to a quotient module $G \cong \mathcal{O}^r/J$, for some submodule $J \subseteq \mathcal{O}^r$. Then, just as for an ideal $J \subseteq \mathcal{O}$, one can consider the module

$$I : J = \{g \in \text{Hom}(\mathcal{O}^r, \mathcal{O}) \mid g(J) \subseteq I\},$$

which thus coincides with the usual colon ideal if $G = \mathcal{O}/J$. In addition, if $J \subseteq \mathcal{O}^r$ is a submodule such that \mathcal{O}^r/J has codimension p , we let $J_{[p]}$ be the intersection of all primary components of J of codimension p . We then obtain the following generalization of (1.13).

Theorem 1.5. *Let G be a finitely generated \mathcal{O} -module of codimension p , and write $G \cong \mathcal{O}^r/J$. If $f = (f_1, \dots, f_p)$ is a regular J -sequence, generating the ideal I , then*

$$(1.14) \quad J_{[p]} = I : (I : J).$$

We note that $I : (I : J)$ is a submodule of $\text{Hom}(\text{Hom}(\mathcal{O}^r, \mathcal{O}), \mathcal{O})$, and that we have a natural isomorphism

$$(1.15) \quad g \in \mathcal{O}^r \xrightarrow{\cong} (\varphi \mapsto \varphi(g)) \in \text{Hom}(\text{Hom}(\mathcal{O}^r, \mathcal{O}), \mathcal{O}).$$

The module $J_{[p]}$ is a submodule of \mathcal{O}^r , and the equality (1.14) is to be understood using this isomorphism (1.15).

We obtain Theorem 1.5 as a rather easy consequence of non-degeneracy in the first argument in the pairing in Theorem 1.1. In fact, one can also go the other way, and (1.13) or (1.14) imply non-degeneracy in the first argument in Theorem 1.1 in the corresponding cases, cf., Lemma 4.8.

1.1. A cohomological residue pairing. We now turn to a somewhat different formulation of the duality theorem. If $G = \mathcal{O}/I$, where I is a complete intersection ideal of codimension n , then in [GH, p. 693], a pairing

$$(1.16) \quad \mathcal{O}/I \times \text{Ext}^n(\mathcal{O}/I, \Omega^n) \rightarrow \mathbb{C}$$

is defined, which is non-degenerate in both arguments. We first recall how this is defined. First of all, we represent $\text{Ext}^n(\mathcal{O}/I, \Omega^n)$ as $H^n(\text{Hom}(K_\bullet, \Omega^n))$, where $(K, \psi) = (\Lambda \mathcal{O}^n, \delta_f)$ is the Koszul complex of f , and an element $\omega \in H^n(\text{Hom}(K_\bullet, \Omega^n))$ can thus be represented as $[\omega_0(e_1 \wedge \dots \wedge e_n)^*]$, where ω_0 is a holomorphic $(n, 0)$ -form, see Section 3 for notation. The pairing (1.16) is then defined by

$$(1.17) \quad (g, \omega) \mapsto \frac{1}{(2\pi i)^n} \int_{\{\cap_{i=1}^n |f_i| = \epsilon\}} \frac{g\omega_0}{f_1 \dots f_n},$$

where ϵ is chosen small enough such that f_1, \dots, f_n and ω_0 are holomorphic on $D_\epsilon := \{\cap_{i=1}^n |f_i| \leq \epsilon\}$ and (f_1, \dots, f_n) has an isolated common zero at $\{0\}$ in D_ϵ , cf., [GH, Chapter 5.1]. This pairing is essentially canonical, and does not depend on the choice of generators f of I .

Passare constructed in [P] a generalization of (1.17), to the case of complete intersection ideals of arbitrary codimension, although the viewpoint of it as a canonical pairing similar to (1.16) was not elaborated. In order to describe this construction, we let $Z \subseteq (\mathbb{C}^n, 0)$ be a subvariety of pure codimension p , and we then define $H_{Z_c}^{p,q}$ to be the module of germs at 0 of smooth (p, q) -forms with compact support, such that they are $\bar{\partial}$ -closed in a neighbourhood of Z .

By [GH, p. 651–655] and Stokes' theorem, there exists a $(0, n-1)$ -form B_f (being essentially like the pullback of the Bochner-Martinelli kernel by

f), such that if χ is a cut-off function which is $\equiv 1$ in a neighbourhood of $\|f\| = \epsilon$, then the right-hand side of (1.17) equals

$$\int g\omega_0 \wedge B_f \wedge \bar{\partial}\chi.$$

More generally, one then gets a pairing

$$\mathcal{O}/I \times \text{Ext}^n(\mathcal{O}/I, \mathcal{O}) \rightarrow \text{Hom}_{\mathbb{C}}(H_{\{0\}^c}^{n,0}, \mathbb{C}),$$

which if $\beta \in H_{\{0\}^c}^{n,0}$ and $\xi = [\xi_0 e^*] \in \text{Ext}^n(\mathcal{O}/I, \mathcal{O})$, then the pairing is given by

$$(1.18) \quad \langle g, \xi \rangle_{GH}(\beta) := \int g\xi_0 B_f \wedge \bar{\partial}\beta,$$

and by taking $\beta = \chi dz_1 \wedge \cdots \wedge dz_n$, one obtains that the pairing is non-degenerate.

In [P], Passare then showed that for $I = J(f_1, \dots, f_p)$ a complete intersection ideal of codimension $p \leq n$, one can define a $(0, p-1)$ -form B_f similar to above, such that if $g \in \mathcal{O}$, then

$$(1.19) \quad g \in I \text{ if and only if } \int g B_f \wedge \bar{\partial}\beta = 0 \text{ for all } \beta \in H_{Z(I)^c}^{n, n-p}.$$

With the help of (1.19), and using a similar construction as in [GH], one then obtains a non-degenerate pairing

$$(1.20) \quad \mathcal{O}/I \times \text{Ext}^p(\mathcal{O}/I, \mathcal{O}) \rightarrow \text{Hom}_{\mathbb{C}}(H_{Z(I)^c}^{n, n-p}, \mathbb{C}).$$

Inspired by the construction of residue currents of Andersson and Wulcan, Lundqvist generalized in [L2] (1.19) to pure dimension ideals, and one can also use this construction obtain a pairing (1.20), which is described in Section 7.1.

We now compare these pairings with the pairing from Theorem 1.1 and Theorem 1.3. A local cohomology class $T \in H_Z^p(\mathcal{O})$ can be represented by a $\bar{\partial}$ -closed $(0, p)$ -current μ with support on Z , modulo $\bar{\partial}$ of $(0, p-1)$ -currents with support on Z , see (3.6). One then obtains a map

$$(1.21) \quad R : H_Z^p(\mathcal{O}) \rightarrow \text{Hom}_{\mathbb{C}}(H_{Z^c}^{n, n-p}, \mathbb{C}),$$

where $R(T) \in \text{Hom}_{\mathbb{C}}(H_{Z^c}^{n, n-p}, \mathbb{C})$ is given as $R(T)(\gamma) = \int \mu \wedge \gamma$. This exists since $\gamma \wedge \mu$ is a (n, n) -current with compact support, and by the fact that γ is $\bar{\partial}$ -closed near Z , it follows easily that $R(T)$ so defined is independent of the choice of representative μ of T .

Theorem 1.6. *Let G be a finitely generated \mathcal{O} -module, and let $Z \supseteq \text{supp } G_{(p)}$, where Z has pure codimension p . There exist a non-degenerate canonical explicitly given pairing*

$$(1.22) \quad G^{(p)} \times \text{Ext}^p(G, \mathcal{O})^{(p)} \rightarrow \text{Hom}_{\mathbb{C}}(H_{Z^c}^{n, n-p}, \mathbb{C}),$$

which is functorial in G . The pairing (1.22) can be defined by composing the pairing (1.9) with (1.21), and if G has codimension $\geq p$, then this pairing coincides with (7.4).

Since the pairing in (1.22) can be defined in terms of the pairing (1.9), non-degeneracy in Theorem 1.6 implies non-degeneracy in Theorem 1.3. The pairing (7.4) is the pairing defined by Lundqvist. By the main result of [L2], if $G = \mathcal{O}/J$ has pure codimension p , then the pairing (7.4) is non-degenerate in the first argument, which thus implies the non-degeneracy in the first argument in Theorem 1.1 in this case. In fact, one can also go the other way around, and non-degeneracy in Theorem 1.3 implies non-degeneracy also in Theorem 1.6, see Lemma 7.6.

We also mention that a variant of such residues was considered also by Lejeune-Jalabert, see [LJ1] and [LJ2], although the main purpose of these articles was to obtain explicit representations of the fundamental cycle of Cohen-Macaulay ideals. Especially in the Artinian case, she obtained explicit expressions for such residues as integrals, which we by functoriality of the pairing of Lundqvist (Proposition 7.2), can see that these definitions indeed coincide, see Example 7.8. In [LJ1] and [LJ2], non-degeneracy of the pairing was not considered.

1.2. Structure of the proof of Theorem 1.1 and Theorem 1.3. The description of the pairings in Theorem 1.1 and Theorem 1.3, and the proof of these theorems and Theorem 1.2 occupy the majority of the article, and is divided into several parts. In addition, for some of the statements, we give several different proofs, so we briefly outline here the disposition of these proofs.

First of all, we give two different ways of defining the pairings (1.2) and (1.8), which are given either algebraically by (2.9), or analytically by (6.9). That these pairings are both functorial is proven in Proposition 2.6 and Proposition 6.3. That these pairings then descend to the pairings (1.3) and (1.9) then follow by a general result about pairings of this form, Corollary 2.4. That these two pairings coincide is proven in Proposition 6.5.

Regarding non-degeneracy, it is proven for the pairing (2.9) in Proposition 5.7 and Proposition 5.9, and for the pairing (6.9) in Proposition 6.9. We also give an alternative proof of non-degeneracy of the pairing (2.9) for $G = \mathcal{O}/J$, where J has codimension $\geq p$ in Section 4.1, see Lemma 4.4 and Lemma 4.8.

Finally, the surjectivity and non-surjectivity of the induced morphisms (1.10) and (1.11) is only proven for the pairing (2.9) in Proposition 5.11, Proposition 5.12 and Proposition 5.13. The slightly differently formulated special case (1.7) of (1.11) is proven in Corollary 5.8.

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2. THE GROTHENDIECK PAIRING

We begin by recalling the definition of local cohomology, and the statement of the local duality theorem from [H], in order to compare with our

theorems, and as well describe the first of the definitions of the pairings (1.2) and (1.8).

2.1. Local cohomology and the Grothendieck pairing. If R is a Noetherian commutative ring, $J \subseteq R$ an ideal, and G an R -module, we define the j -th local cohomology module $H_J^j(G)$ of G with support in J as

$$H_J^j(G) := \varinjlim_t \operatorname{Ext}^j(\mathcal{O}/J^t, G).$$

A rather extensive treatment of local cohomology and applications can be found in [ILL⁺]. We remark that the local cohomology modules are not the same as the germs of the local cohomology sheaves, see Remark 3.3.

As the local cohomology modules $H_J^k(G)$ only depend on the radical of J , see [ILL⁺, Proposition 7.3 and Theorem 7.8], if $Z \subseteq (\mathbb{C}^n, 0)$ is a germ of an analytic subvariety of \mathbb{C}^n , and $J \subseteq \mathcal{O}$ is an ideal such that $Z = Z(J)$, then we write the corresponding local cohomology as

$$(2.1) \quad H_Z^j(G) := H_J^j(G) = \varinjlim_t \operatorname{Ext}^j(\mathcal{O}/J^t, G).$$

If $\operatorname{supp} G \subseteq Z$, then for $t \gg 1$, $J^t \subseteq \operatorname{ann} G$, so $H_Z^0(G) \cong G$. If Z has codimension p , and $\operatorname{supp} G_{(p)} \subseteq Z$, then $H_Z^0(G) = G_{(p)}$.

Inspired by Serre's duality theorem for smooth projective varieties, Grothendieck obtained the following local duality theorem, formulated in terms of local cohomology, [H, Theorem 6.3]. Note that in comparison to Theorem 1.3 (and Theorem 1.1 and Theorem 1.2), we allow local cohomology with support in more general varieties than just $Z = \{0\}$, but on the other hand, we only consider the ring $R = \mathcal{O}$, and only the case $i = 0$.

Theorem 2.1. *Let R be a Gorenstein ring, G a finitely generated R -module, and \mathfrak{m} be the maximal ideal in R . Then there is a natural pairing*

$$(2.2) \quad H_{\mathfrak{m}}^i(G) \times \operatorname{Ext}^{n-i}(G, R) \rightarrow H_{\mathfrak{m}}^n(R),$$

inducing isomorphisms

$$(2.3) \quad H_{\mathfrak{m}}^i(G) \cong \operatorname{Hom}(\operatorname{Ext}^{n-i}(G, R), H_{\mathfrak{m}}^n(R)) \text{ and}$$

$$(2.4) \quad \operatorname{Ext}^{n-i}(G, R)^\wedge \cong \operatorname{Hom}(H_{\mathfrak{m}}^i(G), H_{\mathfrak{m}}^n(R)),$$

where $^\wedge$ denotes completion with respect to the \mathfrak{m} -adic topology.

Remark 2.2. Note also that if we compare the second isomorphism in (2.4) with (1.11) in the case $i = 0$, $p = n$, then one takes the completion in (2.4), but since $\operatorname{Ext}^n(G, \mathcal{O})$ is Artinian, see Proposition 2.3 below, its completion with respect to \mathfrak{m} is the module itself. In addition, since $G_{(n)} = H_{\mathfrak{m}}^0(G)$ and $\operatorname{Ext}^n(G, \mathcal{O})$ are Artinian, they are Cohen-Macaulay, and hence (1.10) and (1.11) are both isomorphism, and then coincide with the isomorphisms (2.3) and (2.4), so Theorem 2.1 and Theorem 1.3 coincide when $R = \mathcal{O}$, $i = 0$ and $p = n$.

We elaborate here a bit on the definition of the pairing in (2.2). The definition of the pairing (2.2) as described in [H, Chapter 6] works equally well more generally to give a pairing

$$(2.5) \quad H_Z^i(G) \times \operatorname{Ext}^{p-i}(G, \mathcal{O}) \rightarrow H_Z^p(\mathcal{O}).$$

In general, the pairing (2.5) is defined in terms of the so-called Yoneda pairing of Ext , which is a pairing of the form

$$\text{Ext}^i(A, B) \times \text{Ext}^j(B, C) \rightarrow \text{Ext}^{i+j}(A, C),$$

and which is described in [H, Chapter 6.1]. Then, the pairing (2.5) is defined as follows. First of all, an element $g \in H_Z^i(G)$ can be represented as an element $g_0 \in \text{Ext}^i(\mathcal{O}/J^t, G)$ by (2.1). Taking the Yoneda pairing with an element $\xi \in \text{Ext}^{p-i}(G, \mathcal{O})$, we obtain an element $(g_0, \xi)_Y \in \text{Ext}^p(\mathcal{O}/J^t, \mathcal{O})$. The desired element is then obtained by composing with the map

$$(2.6) \quad \pi_t : \text{Ext}^p(\mathcal{O}/J^t, \mathcal{O}) \rightarrow \varinjlim_s \text{Ext}^p(\mathcal{O}/J^s, \mathcal{O}) \cong H_Z^p(\mathcal{O}).$$

Using the notation from above, the pairing (2.5) is defined as

$$(2.7) \quad \langle g, \xi \rangle := \pi_t(g_0, \xi)_Y.$$

2.2. Codimension of Ext and local cohomology. In order to obtain the results that the pairings in Theorem 1.1 and Theorem 1.3 descend, we will use the following results about the codimension of Ext -groups and local cohomology groups.

Proposition 2.3. *a) Let G be a finitely generated \mathcal{O} -module. Then $\text{Ext}^p(G, \mathcal{O})$ has codimension $\geq p$. If G has codimension p , then $\text{Ext}^p(G, \mathcal{O})$ has pure codimension p .
b) Let $Z \subseteq (\mathbb{C}^n, 0)$ be a subvariety of pure codimension p . Then $H_Z^p(\mathcal{O})$ has pure codimension p .*

Proof. Part a) is part of [EHV, Theorem 1.1]. Part b) is then a consequence of a) as follows: If P is an associated prime of $H_Z^p(\mathcal{O})$, then there exists some $\mu \in H_Z^p(\mathcal{O})$ such that $\text{ann } \mu = P$. By (2.1), if $t \gg 1$, there exists some $\mu_t \in \text{Ext}^p(\mathcal{O}/J^t, \mathcal{O})$ representing μ , where J is an ideal such that $Z(J) = Z$. By the Noetherianness of \mathcal{O} , we can assume that $t \gg 1$ is such that $\text{ann } \mu_t = P$. Hence, by a), $\text{codim } P = p$. \square

Corollary 2.4. *Assume that A and B are \mathcal{O} -modules of codimension $\geq p$, and let Z be a subvariety of pure codimension p . Then any \mathcal{O} -linear pairing*

$$A \times B \rightarrow H_Z^p(\mathcal{O})$$

descends to a pairing

$$A/A_{(p+1)} \times B/B_{(p+1)} \rightarrow H_Z^p(\mathcal{O}).$$

Proof. Clearly, for any \mathcal{O} -linear pairing, $\text{supp}(a, b) \subseteq (\text{supp } a) \cap (\text{supp } b)$. Thus, if the support of either a or b have codimension $\geq p+1$, then also $\langle a, b \rangle$ has, and thus is 0, since $H_Z^p(\mathcal{O})$ has pure codimension p by Proposition 2.3. \square

As a consequence, any way of defining the pairings (1.2) and (1.8) will descend to pairings (1.3) and (1.9).

2.3. The comparison morphism. We will need the following result about how a morphism of modules induce a morphism of free resolutions of the modules.

Proposition 2.5. *Let $\alpha : F \rightarrow G$ be a homomorphism of finitely generated \mathcal{O} -modules, and let (K, ψ) and (E, φ) be free resolutions of F and G . Then, there exists a morphism $a : (K, \psi) \rightarrow (E, \varphi)$ of complexes which extends α .*

If b is any other such morphism, then there exists a homotopy $s : (K, \psi) \rightarrow (E, \varphi)$ of degree -1 , i.e., consisting of morphisms $s_k : K_k \rightarrow E_{k+1}$, such that $a_i - b_i = \varphi_{i+1}s_i - s_{i-1}\psi_i$.

We say that a extends α if the map induced by a_0 on $K_0/(\text{im } \psi_1) \cong F \rightarrow G \cong E_0/(\text{im } \varphi_1)$ equals α . Both the existence and uniqueness up to homotopy of a follows from defining a or s inductively by a relatively straightforward diagram chase, see [E, Proposition A3.13]. In Example 3.4 below, we give examples of when such a morphism can be explicitly constructed for certain morphism between Koszul complexes.

Note in particular, that if one represents $\text{Ext}^p(F, \mathcal{O})$ and $\text{Ext}^p(G, \mathcal{O})$ as $H^p(\text{Hom}(K_\bullet, \mathcal{O}))$ and $H^p(\text{Hom}(E_\bullet, \mathcal{O}))$, then $\alpha^* : \text{Ext}^p(F, \mathcal{O}) \rightarrow \text{Ext}^p(G, \mathcal{O})$ is given by a_p^* .

2.4. Definition and properties of the pairing. In this section, we give the first possible definition of the pairings (1.2) and (1.8). This definition of the pairing coincides with the pairing (2.5), but since we only consider a special case, i.e., when $i = 0$, we can describe the pairing more concretely. Let Z be such that $\text{supp } G_{(p)} \subseteq Z$, and let J be an ideal such that $Z(J) = Z$. Note that since $\text{supp } G_{(p)} \subseteq Z$, by the Nullstellensatz, if $g \in G_{(p)}$, then $J^t g = 0$ for $t \gg 1$. Thus, any element $g \in G_{(p)}$ defines a morphism

$$(2.8) \quad \epsilon_g : \mathcal{O}/J^t \rightarrow G \text{ such that } \epsilon_g(1) = g.$$

We thus get an induced morphism $\epsilon_g^* : \text{Ext}^p(G, \mathcal{O}) \rightarrow \text{Ext}^p(\mathcal{O}/J^t, \mathcal{O})$. We then compose this with the morphism (2.6). Using the notation from above, the *Grothendieck pairing*

$$G_{(p)} \times \text{Ext}^p(G, \mathcal{O}) \rightarrow H_Z^p(\mathcal{O})$$

is defined as

$$(2.9) \quad \langle g, \xi \rangle_{Gr} := \pi_t(\epsilon_g^* \xi).$$

Proposition 2.6. *The Grothendieck pairing (2.9) is functorial in G .*

Proof. This follows directly from the functoriality of $\text{Ext}^p(\bullet, \mathcal{O})$, since if $f = \alpha(g)$, then $\epsilon_f = \alpha \epsilon_g$, and thus, $\epsilon_g^* = \epsilon_f^* \alpha^*$. \square

3. COMPLETE INTERSECTION IDEALS

In this section, we consider the (already well-known) case of Theorem 1.1, when $G = \mathcal{O}/J$, where J is a complete intersection ideal of codimension p . In Theorem 1.1, we also take $Z = Z(J)$, and take $J(g_1, \dots, g_p)$ as the defining ideal of Z . In this case, $JG = 0$, so for $g \in \mathcal{O}/J$, we can take $t = 1$ in defining the morphism (2.8), i.e., $\epsilon_g : \mathcal{O}/J \rightarrow \mathcal{O}/J$ is just multiplication with g .

Note that when $\epsilon_g : \mathcal{O}/J \rightarrow \mathcal{O}/J$ is just multiplication with g , then the induced pairing $\epsilon_g^* : \text{Ext}^p(\mathcal{O}/J, \mathcal{O}) \rightarrow \text{Ext}^p(\mathcal{O}/J, \mathcal{O})$ can be taken as just multiplication with g . Thus, the pairing in (2.9) is given by

$$(3.1) \quad \langle g, \xi \rangle = \pi_1(g\xi),$$

where $\pi_1 : \text{Ext}^p(\mathcal{O}/J, \mathcal{O}) \rightarrow H_Z^p(\mathcal{O})$.

In order to prove non-degeneracy of the pairing, we use the following result about $\text{Ext}^p(\mathcal{O}/J, \mathcal{O})$, which follows from [GH, Proposition, p. 690].

Lemma 3.1. *Let J be a complete intersection ideal of codimension p . Then*

$$(3.2) \quad \mathcal{O}/J \rightarrow \text{Ext}^p(\mathcal{O}/J, \mathcal{O}),$$

is an isomorphism, and if $I \subseteq J$ is also a complete intersection ideal of codimension p , then the morphism

$$\text{Ext}^p(\mathcal{O}/J, \mathcal{O}) \rightarrow \text{Ext}^p(\mathcal{O}/I, \mathcal{O})$$

induced by the natural surjection $\mathcal{O}/I \rightarrow \mathcal{O}/J$ is injective.

For future reference, we also make the isomorphism (3.2) explicit. Since J is a complete intersection ideal, the Koszul complex $(\bigwedge \mathcal{O}^p, \delta_g)$ of g , which we will denote (K, ψ) , is a free resolution of \mathcal{O}/J , where we denote e_1, \dots, e_p the standard basis for \mathcal{O}^p , such that δ_g is the contraction with $\sum g_i e_i^*$. We can thus represent $\text{Ext}^p(\mathcal{O}/J, \mathcal{O})$ as $H^p(\text{Hom}(K_\bullet, \mathcal{O}))$. The element $e_1 \wedge \dots \wedge e_p$ is a basis of K_p , and the isomorphism (3.2) using this representation of Ext is given by

$$(3.3) \quad h \mapsto h(e_1 \wedge \dots \wedge e_p)^*.$$

Lemma 3.2. *Let $G = \mathcal{O}/J$, where J is a complete intersection ideal of codimension p , and let $Z = Z(J)$. Then the pairing (1.2),*

$$\mathcal{O}/J \times \text{Ext}^p(\mathcal{O}/J, \mathcal{O}) \rightarrow H_Z^p(\mathcal{O}),$$

as given by (2.9), is non-degenerate in both arguments.

Proof. We claim that $\pi_1 : \text{Ext}^p(\mathcal{O}/J, \mathcal{O}) \rightarrow H_Z^p(\mathcal{O})$ is injective, and thus, it is enough to prove non-degeneracy of the pairing

$$\mathcal{O}/J \times \text{Ext}^p(\mathcal{O}/J, \mathcal{O}) \rightarrow \text{Ext}^p(\mathcal{O}/J, \mathcal{O}),$$

which is given by $(g, \xi) \mapsto g\xi$. The fact that this pairing is non-degenerate follows easily from the isomorphism (3.2).

One way of proving the claim is that in order to define $H_Z^p(\mathcal{O})$, if J_t is a family of ideals such that for any t , there exists s and r such that $J^s \subseteq J_r \subseteq J^t$, then

$$(3.4) \quad H_Z^p(\mathcal{O}) \cong \varinjlim_t \text{Ext}^p(\mathcal{O}/J_t, \mathcal{O}),$$

see [ILL⁺, Remark 7.9]. If we let $J_t := J(g_1^t, \dots, g_p^t)$, then $J_t \subseteq J^t$. In addition, by the pigeonhole principle, $J^{pt} \subseteq J_t$. Thus, we can represent $H_Z^p(\mathcal{O})$ using (3.4). Since $J_t \subseteq J$ is a complete intersection ideal of codimension p , the induced map

$$\text{Ext}^p(\mathcal{O}/J, \mathcal{O}) \rightarrow \text{Ext}^p(\mathcal{O}/J_t, \mathcal{O})$$

is injective by Lemma 3.1, so $\pi_1 : \text{Ext}^p(\mathcal{O}/J, \mathcal{O}) \rightarrow H_Z^p(\mathcal{O})$ is injective, proving the claim.

Another less direct way to prove the claim is to instead use Lemma 4.5 below. \square

3.1. Coleff-Herrera products. If $f \in \mathcal{O}$, the *principal value current* $1/f$, can be defined by

$$\frac{1}{f} := \lim_{\epsilon \rightarrow 0^+} \frac{\bar{f}}{|f|^2 + \epsilon},$$

and satisfies $f(1/f) = 1$. Using regularity of the $\bar{\partial}$ -operator on distributions, it is then easily seen that $\text{ann}_{\mathcal{O}} \bar{\partial}(1/f) = J(f)$, i.e., $g \in \mathcal{O}$ lies in the annihilator of $\bar{\partial}(1/f)$, i.e., $g\bar{\partial}(1/f) = 0$, if and only if g belongs to the principal ideal $J(f)$ generated by f . If $f = (f_1, \dots, f_p)$ is a tuple of holomorphic functions defining a complete intersection ideal of codimension p , then Coleff and Herrera showed in [CH] that one can give a reasonable meaning to products of residue currents $\bar{\partial}(1/f_i)$, nowadays called the *Coleff-Herrera product* of f , and written

$$\bar{\partial} \frac{1}{f_p} \wedge \dots \wedge \bar{\partial} \frac{1}{f_1}.$$

The *duality theorem for Coleff-Herrera products*, proven independently by Passare, [P], and Dickenstein-Sessa, [DS1], says that if f defines a complete intersection ideal of codimension p , then

$$(3.5) \quad \text{ann}_{\mathcal{O}} \bar{\partial} \frac{1}{f_p} \wedge \dots \wedge \bar{\partial} \frac{1}{f_1} = J(f_1, \dots, f_p).$$

3.2. Representations of local cohomology classes as currents. If $J \subseteq \mathcal{O}$ is an ideal, $\text{Ext}^p(\mathcal{O}/J, \mathcal{O})$ can be represented as $H^p(\text{Hom}(\mathcal{O}/J, L_{\bullet}))$, where L is an injective resolution of \mathcal{O} . Since the Dolbeault complex $(C^{0,\bullet}, \bar{\partial})$ of $(0, *)$ -currents is an injective resolution of \mathcal{O} , we can thus represent objects in $\text{Ext}^p(\mathcal{O}/J, \mathcal{O})$ as $\bar{\partial}$ -closed $(0, p)$ -currents annihilated by J .

In addition, if one represents $\text{Ext}^p(\mathcal{O}/J^t, \mathcal{O})$ as $H^p(\text{Hom}(\mathcal{O}/J^t, C^{0,\bullet}))$, then the morphism $\text{Ext}^p(\mathcal{O}/J, \mathcal{O}) \rightarrow \text{Ext}^p(\mathcal{O}/J^t, \mathcal{O})$ is induced by the inclusion $\text{Hom}(\mathcal{O}/J, C^{0,p}) \rightarrow \text{Hom}(\mathcal{O}/J^t, C^{0,p})$, which just corresponds to the fact that currents annihilated by J are also annihilated by J^t . Thus, using this representation, any element in $H_Z^p(\mathcal{O})$ can be represented by a $(0, p)$ -current annihilated by J^t for $t \gg 1$, which due to the fact that a current has locally finite order is equivalent to that it has support on $Z = Z(J)$. Thus, one has the following representation of the local cohomology groups,

$$(3.6) \quad H_Z^p(\mathcal{O}) \cong H^p(\text{Hom}(C_Z^{0,\bullet})),$$

where $(C_Z^{0,\bullet}, \bar{\partial})$ is the Dolbeault complex of $(0, *)$ -currents with support on Z .

Remark 3.3. The local cohomology groups we consider are in the local ring $\mathcal{O} = \mathcal{O}_{\mathbb{C}^n, 0}$, which correspond to the stalks of the *moderate cohomology sheaf*, which is what is mainly treated in for example [DS1]. We remark however that these stalks are not the same as the stalks of the *local cohomology sheaf*, cf., the introduction of [DS1].

Another way of representing $\text{Ext}^p(\mathcal{O}/J, \mathcal{O})$ is as elements in $H^p(\text{Hom}(E_\bullet, \mathcal{O}))$, where (E, φ) is a free resolution of \mathcal{O}/J , and by standard homological algebra, there is a canonical isomorphism

$$(3.7) \quad H^p(\text{Hom}(E_\bullet, \mathcal{O})) \cong H^p(\text{Hom}(\mathcal{O}/J, L_\bullet)).$$

If we in particular consider the case when J is a complete intersection ideal of codimension p as above, generated by f_1, \dots, f_p , then one can take the Koszul complex (K, ψ) of f as a free resolution of \mathcal{O}/J , and one has the representation (3.3) of $\text{Ext}^p(\mathcal{O}/J, \mathcal{O})$. One thus gets a canonical isomorphism

$$(3.8) \quad H^p(\text{Hom}(K_\bullet, \mathcal{O})) \cong H^p(\text{Hom}(\mathcal{O}/J, C^{0,\bullet})).$$

In [DS1], the canonical isomorphism (3.8) is expressed in terms of the Coleff-Herrera product, and is given by

$$(3.9) \quad [\xi_0] = [h(e_1 \wedge \dots \wedge e_p)^*] \mapsto \left[h \bar{\partial} \frac{1}{f_p} \wedge \dots \wedge \bar{\partial} \frac{1}{f_1} \right],$$

see the proof of [DS1, Proposition 3.5].

Thus, when J is a complete intersection ideal, and $G = \mathcal{O}/J$, then using the representation (3.3) of $\text{Ext}^p(\mathcal{O}/J, \mathcal{O})$, and the representation (3.6) of $H_Z^p(\mathcal{O})$, the pairing (2.9) is given by

$$(3.10) \quad \langle g, h(e_1 \wedge \dots \wedge e_p)^* \rangle = gh \bar{\partial} \frac{1}{f_p} \wedge \dots \wedge \bar{\partial} \frac{1}{f_1}.$$

Example 3.4. If $I = J(f_1, \dots, f_p)$, and $J = J(g_1, \dots, g_p)$ are two complete intersection ideals of codimension p , then the fact that $I \subseteq J$ is equivalent to that there exists a holomorphic $p \times p$ -matrix A such that $f = Ag$. The fact that $I \subseteq J$ means that one has the natural surjection $\pi : \mathcal{O}/I \rightarrow \mathcal{O}/J$. If one lets $(E, \varphi) = (\bigwedge \mathcal{O}^p, \delta_g)$ and $(K, \psi) = (\bigwedge \mathcal{O}^p, \delta_f)$ be the Koszul complexes of (g_1, \dots, g_p) and (f_1, \dots, f_p) respectively, then it is straightforward to verify that one choice of the morphism $a : (K, \psi) \rightarrow (E, \varphi)$ is $a_k := \bigwedge^k A : \bigwedge^k \mathcal{O}^p \rightarrow \bigwedge^k \mathcal{O}^p$. Hence, using the representation (3.3) of $\text{Ext}^p(\mathcal{O}/I, \mathcal{O})$ and $\text{Ext}^p(\mathcal{O}/J, \mathcal{O})$, the morphism $\text{Ext}^p(\mathcal{O}/J, \mathcal{O}) \rightarrow \text{Ext}^p(\mathcal{O}/I, \mathcal{O})$ induced by $\pi : \mathcal{O}/I \rightarrow \mathcal{O}/J$ is given by multiplication with $a_p = \det A$.

Using the functoriality of the pairing (2.9), and combining this with the expression (3.10) for the pairing in the particular case when $g = h = 1$, one gets that

$$(3.11) \quad \bar{\partial} \frac{1}{g_p} \wedge \dots \wedge \bar{\partial} \frac{1}{g_1} = (\det A) \bar{\partial} \frac{1}{f_p} \wedge \dots \wedge \bar{\partial} \frac{1}{f_1}$$

as cohomology classes. Indeed, the *transformation law* for Coleff-Herrera products as proven in [DS1] or [DS2] states that this holds even as currents.

4. REDUCTION TO THE COMPLETE INTERSECTION CASE AND LINKAGE

It is well-known that for any ideal J of codimension p , one can find a complete intersection ideal $I = J(f_1, \dots, f_p)$ of codimension p contained in J , for a proof, see for example [L4, Lemma 19]. As a consequence of this well-known fact, we have the following generalization, which we will make use of in order to reduce properties for the pairing in Theorem 1.1 to the complete intersection case in the previous section.

Lemma 4.1. *Let G be a finitely generated \mathcal{O} -module. Then, there exists a complete intersection ideal I of codimension p , and a morphism $\alpha : (\mathcal{O}/I)^r \rightarrow G$ for some $r \in \mathbb{N}$, which is surjective onto $G_{(p)}$.*

We will use this to give a rather elementary proof of the non-degeneracy in Theorem 1.1 when G is of the form $G = \mathcal{O}/J$, where J is an ideal of codimension $\geq p$ by means of the theory of linkage. We will also use the non-degeneracy in Theorem 1.1 to prove Theorem 1.5.

Proof. Since $\text{ann } G_{(p)}$ has codimension p , as explained above, we can then find a complete intersection ideal I of codimension p contained in $\text{ann } G_{(p)}$. Since $G_{(p)}$ is finitely generated, there exists a surjective morphism $\pi : \mathcal{O}^r \rightarrow G_{(p)}$ for some $r \in \mathbb{N}$. Since $I \subseteq \text{ann } G_{(p)}$, π induces the surjective morphism $\alpha' : (\mathcal{O}/I)^r \rightarrow G_{(p)}$, and composing this with the inclusion $G_{(p)} \subseteq G$, we obtain the desired morphism α . \square

The following result about vanishing of Ext , follows from [E, Proposition 18.4], and we will use both in this section, in the partial proof of Theorem 1.1 and it will also be an important part in the full proof of Theorem 1.3 in Section 5.

Proposition 4.2. *Let G be a finitely generated \mathcal{O} -module of codimension p . Then $\text{Ext}^r(G, \mathcal{O}) = 0$ for $r < p$.*

Proposition 18.4 in [E] states more generally that $\text{depth}(\text{ann}(M), N)$ is the smallest $r \in \mathbb{N}$ such that $\text{Ext}^r(M, N) \neq 0$. When $N = \mathcal{O}$, which is Cohen-Macaulay, then $\text{depth}(\text{ann}(M), \mathcal{O}) = \text{codim } M$, and thus, Proposition 4.2 follows from in [E, Proposition 18.4].

Lemma 4.3. *Let $Z \subseteq W$ be two subvarieties of $(\mathbb{C}^n, 0)$ of pure codimension p . Then the induced map*

$$(4.1) \quad H_Z^p(\mathcal{O}) \rightarrow H_W^p(\mathcal{O})$$

is injective.

Proof. We let J and I be ideals defining Z and W respectively. By the Nullstellensatz, we can assume that $J \subseteq I$. Then, we have a short exact sequence

$$0 \rightarrow I^t(\mathcal{O}/J^t) \rightarrow \mathcal{O}/J^t \rightarrow \mathcal{O}/I^t \rightarrow 0,$$

and since $I^t(\mathcal{O}/J^t)$ has codimension $\geq p$, we get by Proposition 4.2 and the long exact sequence of Ext an injection

$$0 \rightarrow \text{Ext}^p(\mathcal{O}/I^t, \mathcal{O}) \rightarrow \text{Ext}^p(\mathcal{O}/J^t, \mathcal{O}).$$

Similarly, if $s > t$, we have an injection

$$0 \rightarrow \text{Ext}^p(\mathcal{O}/J^t, \mathcal{O}) \rightarrow \text{Ext}^p(\mathcal{O}/J^s, \mathcal{O}),$$

and these two injections together give the injectivity of (4.1). \square

4.1. Partial proofs of Theorem 1.1 and Theorem 1.5. If I is a complete intersection ideal of codimension p , and $G = \mathcal{O}/I$, then we have an explicit expression for the pairing (1.2) as given by (3.1) or by (3.10) depending on how one represents $H_Z^p(\mathcal{O})$.

We can now rather easily obtain non-degeneracy in the second argument for general modules of codimension $\geq p$.

Lemma 4.4. *Let G be a finitely generated \mathcal{O} -module of codimension $\geq p$, and let $Z \supseteq \text{supp } G$ be of pure codimension p . Consider a pairing*

$$G \times \text{Ext}^p(G, \mathcal{O}) \rightarrow H_Z^p(\mathcal{O}),$$

which is functorial in G . If the pairing is non-degenerate in the second argument for G of the form $G = \mathcal{O}/I$, where I is any complete intersection ideal of codimension p , then it is non-degenerate in the second argument for any finitely generated \mathcal{O} -module.

Proof. Note first that if the pairing is non-degenerate for \mathcal{O}/I , then by functoriality, it is also non-degenerate for $F = (\mathcal{O}/I)^r$. Take now $\alpha : (\mathcal{O}/I)^r \rightarrow G$ as in Lemma 4.1. Assume that $\langle g, \xi \rangle = 0$ for all $\xi \in \text{Ext}^p(G, \mathcal{O})$. By functoriality and the non-degeneracy for $(\mathcal{O}/I)^r$, we get that $\alpha^* \xi = 0$.

We have an exact sequence

$$0 \rightarrow H \rightarrow (\mathcal{O}/I)^r \rightarrow G \rightarrow 0,$$

induced by α , where H has codimension $\geq p$. Thus, by the long exact sequence of Ext associated to this short exact sequence, and the fact that $\text{Ext}^{p-1}(H, \mathcal{O}) = 0$ by Proposition 4.2, we get an injection

$$(4.2) \quad \alpha^* : \text{Ext}^p(G, \mathcal{O}) \rightarrow \text{Ext}^p((\mathcal{O}/I)^r, \mathcal{O}).$$

Since $\alpha^* \xi = 0$, we thus conclude that $\xi = 0$. \square

In order to prove non-degeneracy in the first argument of the pairing, we begin with the following lemma, generalizing the argument in the proof of Lemma 3.2.

Lemma 4.5. *Let H be a finitely generated \mathcal{O} -module of codimension $\geq p$, and let $J \subseteq \text{ann } H$ be such that $Z = Z(J)$ has pure codimension p . Then*

$$(4.3) \quad \text{Hom}(H, H_Z^p(\mathcal{O})) \cong \text{Hom}(H, \text{Ext}^p(\mathcal{O}/J, \mathcal{O})).$$

Proof. It is enough to prove that

$$(4.4) \quad \text{Hom}(H, \text{Ext}^p(\mathcal{O}/J, \mathcal{O})) \xrightarrow{\cong} \text{Hom}(H, \text{Ext}^p(\mathcal{O}/J^t, \mathcal{O}))$$

for any $t \geq 1$. Consider for $t \geq 1$ the short exact sequence

$$0 \rightarrow J\mathcal{O}/J^t \rightarrow \mathcal{O}/J^t \rightarrow \mathcal{O}/J \rightarrow 0.$$

By the long exact sequence of Ext , and Proposition 4.2, we have an exact sequence

$$0 \rightarrow \text{Ext}^p(\mathcal{O}/J, \mathcal{O}) \rightarrow \text{Ext}^p(\mathcal{O}/J^t, \mathcal{O}) \rightarrow \text{Ext}^p(J\mathcal{O}/J^t, \mathcal{O}).$$

Hence, by left exactness of $\text{Hom}(H, \bullet)$, (4.4) is injective, and it remains to prove that it is surjective. Consider thus $\beta \in \text{Hom}(H, \text{Ext}^p(\mathcal{O}/J^t, \mathcal{O}))$. If $h \in H$, then $J\beta(h) = 0$. If one represents Ext with the help of an injective resolution in the second argument, one sees that the image of $\beta(h)$

in $\text{Ext}^p(J\mathcal{O}/J^t, \mathcal{O})$ is 0, so $\beta(h)$ lifts to a unique element in $\text{Ext}^p(\mathcal{O}/J, \mathcal{O})$, and β thus lifts to a morphism $\text{Hom}(H, \text{Ext}^p(\mathcal{O}/J, \mathcal{O}))$. \square

If we take $H = \text{Ext}^p(G, \mathcal{O})$ in Lemma 4.5, it is thus enough to prove non-degeneracy in the first argument for the Yoneda pairing, without composing with π_t from (2.6).

We will also use the following alternative description of the module $I : J$ which appears in Theorem 1.5. To begin with, we set the notation which we will use throughout the rest of this section. We assume that G is a finitely generated \mathcal{O} -module of codimension $\geq p$. We let

$$\alpha : (\mathcal{O}/I)^r \rightarrow G$$

be a surjective morphism as in Lemma 4.1, where I is a complete intersection ideal of codimension p contained in $\text{ann } G$. Let (L, δ_f) be the Koszul complex of a minimal set (f_1, \dots, f_p) of generators of I , and we let e_1, \dots, e_p be the standard basis of \mathcal{O}^r , so that $e := e_1 \wedge \dots \wedge e_p$ is a basis element of L_p . We let $(K, \psi) = (L \otimes \mathcal{O}^r, \psi \otimes \text{Id}_{\mathcal{O}^r})$ be the direct sum of r copies of (L, δ_f) , which is a free resolution of $(\mathcal{O}/I)^r$, and finally, we let

$$a : (K, \psi) \rightarrow (E, \varphi)$$

be a morphism of complexes extending α , as in Proposition 2.5.

Lemma 4.6. *Let $G \cong \mathcal{O}^r/J$, where G has codimension $\geq p$, and let (E, φ) , (K, ψ) , (L, δ_f) and $a : (K, \psi) \rightarrow (E, \varphi)$ be as above. Then,*

$$I : JL_p = IK_p^* + (\ker \varphi_{p+1}^*)a_p.$$

Note that if $G = \mathcal{O}/J$ is Cohen-Macaulay of codimension p , and (E, φ) has length p , then $(\ker \varphi_{p+1}^*)a_p$ is simply the ideal $J(a_p)$ generated by the entries of a_p , i.e.,

$$I : J = I + J(a_p).$$

In this case, Lemma 4.6 is well-known, and can be found for example in Lemma 3.2 in [FH]. The following is an adaption of this proof to our more general situation.

Proof. We let $M := (\ker \varphi_{p+1}^*)a_p$. First of all, we prove that $IK_p^* + M \subseteq I : JL_p$. It is clear that $IK_p^* \subseteq I : JL_p$, so we want to prove that $M \subseteq I : JL_p$. Take $g \in J \subseteq \mathcal{O}^r$. The element g induces a morphism $\text{Hom}(\mathcal{O}/I, (\mathcal{O}/I)^r)$, which extends to the morphism of complexes

$$g \otimes \text{Id}_L : L \rightarrow \mathcal{O}^r \otimes L \cong K.$$

Since $g \in J$, we get that $a(g \otimes \text{Id}_L) : (L, \delta_f) \rightarrow (E, \varphi)$ is a morphism of complexes extending the zero morphism $\mathcal{O}/I \rightarrow G$. Thus, by Proposition 2.5, there exists a homotopy $s : (L, \delta_f) \rightarrow (E, \varphi)$ between 0 and $a(g \otimes \text{Id}_L)$. In particular,

$$(4.5) \quad a_p(g \otimes \text{Id}_{L_p}) = s_{p-1}(\delta_f)_p + \varphi_{p+1}s_p.$$

If $\xi \in \ker \varphi_{p+1}^*$, and we apply this to (4.5), we get that

$$\xi a_p(g \otimes \text{Id}_{L_p}) = \xi s_{p-1}(\delta_f)_p \subseteq I,$$

since $\text{im}(\delta_f)_p \subseteq IL_{p-1}$. Hence, $M \subseteq I : JL_p$.

Conversely, we consider an element $\gamma \in I : JL_p$. By the isomorphism $L_p \cong \mathcal{O}$ (given by e^* , the dual of the basis $e = e_1 \wedge \cdots \wedge e_p$ of L_p), we can consider γ as an element $\tilde{\gamma} : \mathcal{O}^r \rightarrow \mathcal{O}$, and we have that

$$(4.6) \quad \gamma = e^*(\tilde{\gamma} \otimes \text{Id}_{L_p}).$$

The morphism $\tilde{\gamma}$ descends to a morphism $\mathcal{O}^r/J \rightarrow \mathcal{O}/I$, and by Proposition 2.5, we can find a morphism of complexes $b : (E, \varphi) \rightarrow (L, \delta_f)$ extending this morphism. The morphism $\tilde{\gamma}$ also induces a morphism $(\mathcal{O}/I)^r \rightarrow (\mathcal{O}/I)$, which in turn induces a morphism of complexes $(K, \psi) \rightarrow (L, \delta_f)$, which is given simply as

$$\tilde{\gamma} \otimes \text{Id}_L : K \cong \mathcal{O}^r \otimes L \rightarrow \mathcal{O} \otimes L \cong L.$$

Then, $\tilde{\gamma} \otimes \text{Id}_L$ and ba both extend the morphism $(\mathcal{O}/I)^r \rightarrow \mathcal{O}/I$ induced by $\tilde{\gamma}$, so by Proposition 2.5, ba is homotopic to $\tilde{\gamma} \otimes \text{Id}_L$, and in particular, there exists s_{p-1} such that

$$(4.7) \quad \tilde{\gamma} \otimes \text{Id}_{L_p} - b_p a_p = s_{p-1} \psi_p.$$

In addition, since $b : (E, \varphi) \rightarrow (L, \delta_f)$ is a morphism of complexes, and $L_{p+1} = 0$, we get that $\varphi_{p+1}^* b_p = 0$. Thus, $e^* b_p \in \ker \varphi_{p+1}^*$. To conclude, applying e^* to (4.7), and using this in combination with (4.6), we get that

$$\gamma = (e^* b_p) a_p + e^* s_{p-1} \psi_p \subseteq M + IK_p^*,$$

and we have proven the other inclusion. \square

In order to prove non-degeneracy in the first argument, one cannot as easily reduce non-degeneracy to the complete intersection case, but using (1.13) from the theory of linkage, we can do this reduction when G is of the form $G = \mathcal{O}/J$, where G has codimension $\geq p$.

In the case of non-pure dimension, we first relate $G^{(p)}$ in Theorem 1.1 and $J_{[p]}$ in (1.13) or (1.14).

Remark 4.7. If $G = \mathcal{O}^r/J$, where G has codimension $\geq p$, then we claim that $(\mathcal{O}^r/J)^{(p)} = \mathcal{O}^r/J_{[p]}$. To see this, note that if $g \in (\mathcal{O}^r/J)_{(p+1)}$, then $g \in J_{[p]}$. Thus, we get a well-defined surjective map $(\mathcal{O}^r/J)^{(p)} \rightarrow \mathcal{O}^r/J_{[p]}$. In addition, it is injective, since if $g = 0$ in $\mathcal{O}^r/J_{[p]}$, and if we write $J = J_{[p]} \cap J_{[\geq p+1]}$, then $g \in J$ outside of $\text{supp } \mathcal{O}^r/J_{[\geq p+1]}$ which has codimension $\geq p+1$, and thus, $g \in (\mathcal{O}^r/J)_{(p+1)}$.

Lemma 4.8. *Let $G = \mathcal{O}/J$, where J has codimension $\geq p$, and let $Z \supseteq \text{supp } G$ be of pure codimension p . Consider a pairing*

$$G \times \text{Ext}^p(G, \mathcal{O}) \rightarrow H_Z^p(\mathcal{O}),$$

which is functorial in G . If the descended pairing

$$G/G_{(p+1)} \times \text{Ext}^p(G, \mathcal{O}) \rightarrow H_Z^p(\mathcal{O}),$$

is non-degenerate in the second argument for G of the form $G = \mathcal{O}/I$, where I is any complete intersection ideal of codimension p , then it is non-degenerate in the first argument for any $G = \mathcal{O}/J$, where J has codimension $\geq p$.

Proof. We let (E, φ) be a free resolution of $G = \mathcal{O}/J$, and using the representation $\text{Ext}^p(G, \mathcal{O}) \cong H^p(\text{Hom}(E_\bullet, \mathcal{O}))$, we write any $\xi \in \text{Ext}^p(G, \mathcal{O})$ as $\xi = [\xi_0]$, where $\xi_0 \in \ker \varphi_{p+1}^*$. We let $I \subseteq J$ be a complete intersection ideal contained in J , (K, ψ) the Koszul complex of a set of minimal generators of I , and let $a : (K, \psi) \rightarrow (E, \varphi)$ be a morphism of complexes extending the natural surjection $\pi : \mathcal{O}/I \rightarrow \mathcal{O}/J$.

Take $g \in G = \mathcal{O}/J$ such that $\langle g, \xi \rangle = 0$ for all ξ . We want to show that $g \in G_{(p+1)}$. For $f \in \mathcal{O}/I$, we also get that $\langle fg, \xi \rangle = 0$ for all ξ . Thus, by \mathcal{O} -linearity, functoriality, and using the representation of Ext above, we get that

$$\langle f, [a_p^* \xi_0 g] \rangle = 0$$

for all $f \in \mathcal{O}/I$, and all $\xi_0 \in \ker \varphi_{p+1}$. By non-degeneracy in the second argument, for \mathcal{O}/I , we get that $\xi_0 a_p g = 0$ in $\text{Ext}^p(\mathcal{O}/I, \mathcal{O})$. Thus, $g \in I : I + (\ker \varphi_{p+1}^*)a_p$, and by Lemma 4.6, $g \in I : (I : J)$, so, $g \in J_{[p]}$ by (1.13), i.e., $g \in G_{(p+1)}$. \square

Arguing in a similar way, one would also obtain that (1.14) implies non-degeneracy in the first argument for any finitely generated \mathcal{O} -module G of codimension $\geq p$, and not just G of the form $G = \mathcal{O}/J$. However, since we know of a proof of (1.14) which does not depend on Theorem 1.1 only in the case when $G = \mathcal{O}/J$, i.e., (1.13), we have only stated Lemma 4.8 in this case.

In the proof of Lemma 4.8, we used Lemma 4.6 and (1.13) to obtain a proof of non-degeneracy in the first argument of the pairing in Theorem 1.1 when G is of the form $G = \mathcal{O}/J$. Here we show that we can also go the other way, and prove (1.13), or more generally (1.14) using the functoriality and non-degeneracy in the first argument of the pairing in Theorem 1.1.

Proof of Theorem 1.5. It is clear that $J_{[p]} \subseteq I : (I : J_{[p]})$. In addition, we claim that $I : J \subseteq I : J_{[p]}$, which implies that $J_{[p]} \subseteq I : (I : J)$. To prove the claim, we first write $J = J_{[p]} \cap J_{[\geq p+1]}$, where $J_{[\geq p+1]}$ is the intersection of all primary components of codimension $\geq p+1$. If $\gamma(J) \subseteq I$, then $\gamma(J_{[p]}) \subseteq I$ outside of $Z(J_{[\geq p+1]})$ which has codimension $\geq p+1$, so $\gamma(J_{[p]}) \subseteq (\mathcal{O}/I)_{(p+1)}$, and thus, $\gamma J_{[p]} = 0$ in \mathcal{O}/I , since \mathcal{O}/I has pure codimension p .

It remains to prove the reverse inclusion $I : (I : J) \subseteq J_{[p]}$. We now take (E, φ) , (K, ψ) , (L, δ_f) and $a : (K, \psi) \rightarrow (E, \varphi)$ as before the statement of Lemma 4.6. Assume now that $g \in I : (I : J)$. A priori, $g \in G$, and we can take a representative in \mathcal{O}^r . This element induces a morphism $\epsilon_g : \text{Hom}(\mathcal{O}/I, (\mathcal{O}/I)^r)$, and we let $c : (L, \delta_f) \rightarrow (K, \psi)$ be the morphism

$$g \otimes \text{Id}_L : L \rightarrow \mathcal{O}^r \otimes L = K$$

which extends ϵ_g . By the fact that $g \in I : (I : J)$, we get that $c_p^* \in (I : (I : JL_p))$. We note that $g = \alpha \epsilon_g(1)$, so by functoriality of the pairing, we get

$$\langle g, \xi \rangle = \langle \alpha(\epsilon_g(1)), \xi \rangle = \langle 1, \epsilon_g^* \alpha^* \xi \rangle.$$

If we represent $\xi = [\xi_0]$, where $\xi_0 \in \ker \varphi_{p+1}^*$, then

$$[\epsilon_g^* \alpha^* \xi_0] = [\xi_0 a_p c_p].$$

Since $c_p^* \in (I : (I : JL_p)) = I : (IK_p^* + (\ker \varphi_{p+1}^*)a_p)$, we get that $\text{im } \xi_0 a_p c_p \in I$, so $[\xi_0 a_p c_p] = 0$, since $I \text{Ext}^p(\mathcal{O}/I, \mathcal{O}) = 0$. Thus, $\langle g, \xi \rangle = 0$ for all ξ , and

by non-degeneracy of the pairing (1.3), we get that $g = 0$ in $G^{(p)}$, i.e., $g \in J_{[p]}$ by Remark 4.7. \square

5. PROOFS OF THEOREM 1.1, THEOREM 1.2 AND THEOREM 1.3

We will use the following consequences of Proposition 4.2.

Corollary 5.1. *Let*

$$A \rightarrow B \rightarrow C \rightarrow 0$$

be an exact sequence of finitely generated \mathcal{O} -modules such that A , B and C have support in a variety of codimension $\geq p$. Then

$$0 \rightarrow \text{Ext}^p(C, \mathcal{O}) \rightarrow \text{Ext}^p(B, \mathcal{O}) \rightarrow \text{Ext}^p(A, \mathcal{O})$$

is exact.

Proof. We let $K = \ker(B \rightarrow C)$, and thus have a short exact sequence

$$0 \rightarrow K \rightarrow B \rightarrow C \rightarrow 0,$$

and since $\text{codim } K \geq p$, we get from the long exact sequence of Ext and Proposition 4.2 an exact sequence

$$(5.1) \quad 0 \rightarrow \text{Ext}^p(C, \mathcal{O}) \rightarrow \text{Ext}^p(B, \mathcal{O}) \rightarrow \text{Ext}^p(K, \mathcal{O}).$$

Since $K = \ker(B \rightarrow C) = \text{im}(A \rightarrow B)$, if we let $A' := \ker(A \rightarrow B)$, we have a short exact sequence

$$0 \rightarrow A' \rightarrow A \rightarrow K \rightarrow 0,$$

and since A' has codimension $\geq p$, we get as above an injection

$$\text{Ext}^p(K, \mathcal{O}) \rightarrow \text{Ext}^p(A, \mathcal{O}).$$

Composing (5.1) with this injection at the end, we get the desired short exact sequence. \square

We will combine Corollary 5.1 with the following result, which is a combination of [H, Lemma 4.4] and [H, Proposition 4.5].

Proposition 5.2. *Let $J \subseteq \mathcal{O}$ be an ideal, and let T be a contravariant additive functor from the category of finitely generated \mathcal{O} -modules with support in $Z(J)$ to the category of abelian groups. Then there is a natural functorial morphism*

$$(5.2) \quad T \rightarrow \text{Hom}_{\mathcal{O}}(\bullet, \varinjlim_t T(\mathcal{O}/J^t)),$$

and this morphism is an isomorphism if and only if T is left exact.

From the proof of Lemma 4.4 and Lemma 4.1 in [H], it is readily verified that if G is a finitely generated \mathcal{O} -module with support in $Z(J)$, then the morphism $T(G) \rightarrow \text{Hom}_{\mathcal{O}}(G, \varinjlim_t T(\mathcal{O}/J^t))$ in (5.2) is given by

$$\xi \in T(G) \mapsto (g \mapsto \pi_t(\epsilon_g^* \xi)),$$

where $t \gg 1$ is such that $J^t g = 0$, $\epsilon_g : \mathcal{O}/J^t \rightarrow G$, $\epsilon_g(1) = g$, and π_t is the natural morphism $\pi_t : T(\mathcal{O}/J^t) \rightarrow \varinjlim_s T(\mathcal{O}/J^s)$. In particular, when $T = \text{Ext}^p(\bullet, \mathcal{O})$, then (5.2) coincides with the morphism induced by the pairing (2.5) when $i = 0$ and G has support in $Z = Z(J)$.

5.1. Singularity subvarieties and the S_k -property. The statements of Theorem 1.2 and Theorem 1.3 involve the so-called Serre's S_2 -property of a finitely generated \mathcal{O} -module G , which can be expressed in terms of the following singularity subvarieties associated to G . Although the S_2 -property is defined for arbitrary finitely generated \mathcal{O} -modules, we will here only treat the case of \mathcal{O} -modules G such that $Z(G)$ has pure codimension p , as the definition becomes easier in this case, and this case is the only one that we will use.

We will here consider the singularity subvarieties and the S_2 property simultaneously for a finitely generated \mathcal{O} -module G , or a coherent analytic sheaf \mathcal{G} on a complex manifold X , since we will rely on a characterization of S_2 in the latter setting. For G a finitely generated \mathcal{O} -module, by taking a free resolution (E, φ) of G defined in some neighbourhood of 0, (E, φ) induces a coherent sheaf $\mathcal{G} = \text{coker } \varphi_1$ on this neighbourhood of 0 such that $\mathcal{G}_0 = G$.

Given a (locally) free resolution (E, φ) of G or \mathcal{G} , the associated singularity subvarieties $Z_\ell = Z_\ell^E$ are defined as the sets where φ_ℓ does not have optimal rank, where $\ell = 1, 2, \dots$. By [E, Corollary 20.12],

$$(5.3) \quad Z_{\ell+1} \subseteq Z_\ell.$$

In addition, by [E, Theorem 20.9] one has

$$(5.4) \quad \text{codim } Z_\ell \geq \ell,$$

and by [E, Corollary 20.14], G has no associated prime of codimension ℓ , if and only if

$$(5.5) \quad \text{codim } Z_\ell \geq \ell + 1.$$

If G or \mathcal{G} is such that $Z = Z(G)$ or $Z = Z(\mathcal{G})$ has pure codimension p , then G or \mathcal{G} is said to be S_k if

$$\text{codim } Z_\ell \geq \ell + k \text{ for } \ell \geq p + 1.$$

Note that if G is a finitely generated \mathcal{O} -module, then G is S_k if and only if the induced coherent sheaf \mathcal{G} is S_k in some neighbourhood of 0. By (5.5), if G has pure codimension, then G is S_1 .

If $G = \mathcal{O}_X/J$, where $J = J_Z$, the ideal of holomorphic functions vanishing on a subvariety $Z \subseteq X$, then by the Serre criterion for normality, Z is normal if and only if G is S_2 and R_1 (where R_1 means that Z_{sing} has codimension at least 2 in Z), cf., for example [M, Theorem 1].

It follows easily from (5.3) that

$$\text{supp Ext}^p(G, \mathcal{O}) \subseteq Z_p(G).$$

Thus, if G does not have any associated primes of codimension p , we get from (5.5) that

$$(5.6) \quad \text{codim Ext}^p(G, \mathcal{O}) \geq p + 1.$$

The following characterization of S_2 is classical, namely that sections extend over subvarieties of codimension ≥ 2 . Here, we begin with the formulation for coherent analytic sheaves, which is a slight reformulation of part of the results in [ST, Theorem 1.14].

Proposition 5.3. *Let \mathcal{G} be a coherent analytic sheaf of pure codimension p on a complex manifold X . Then \mathcal{G} is S_2 if and only if*

$$(5.7) \quad \mathcal{G}(U) \rightarrow \mathcal{G}(U \setminus V)$$

is an isomorphism for any open set $U \subseteq X$, and V a subvariety of U of codimension $\geq p + 2$.

In [ST], certain singularity subvarieties S_k associated to a coherent analytic sheaf are defined, which are related to our singularity subvarieties Z_k by the simple relation that if $\dim X = n$, then $Z_k = S_{n-k}$, see [L4, Proposition 26].

Proof. From [ST, Theorem 1.14], for any subvariety A of X ,

$$(5.8) \quad \text{codim } A \cap Z_k \geq k + 2 \text{ for } k \geq 1$$

if and only if

$$\mathcal{G}(U) \rightarrow \mathcal{G}(U \setminus A)$$

is an isomorphism for any open set $U \subseteq X$. If we take $A = V$ of codimension $\geq p + 2$, then (5.8) holds for $k \leq p$. In addition, if \mathcal{G} is S_2 , then (5.8) holds for $k \geq p + 1$ by definition of S_2 , so (5.7) is an isomorphism.

Conversely, we assume that (5.7) is an isomorphism. Since \mathcal{G} has pure dimension p , $\text{codim } Z_{p+1} \geq p + 2$ by (5.4), so (5.7) is an isomorphism for $V = Z_{p+1}$. Thus, (5.8) holds for $A = Z_{p+1}$. By (5.3), $Z_k \cap Z_{p+1} = Z_k$ for $k \geq p + 1$, so if (5.8) holds with $A = Z_{p+1}$, we conclude that \mathcal{G} is S_2 . \square

We obtain as a corollary of Proposition 5.3 the following variant in the case of finitely generated \mathcal{O} -modules.

Corollary 5.4. *Let F, G and H be finitely generated \mathcal{O} -modules, and assume that there is an exact sequence*

$$0 \rightarrow F \xrightarrow{\alpha} G \rightarrow H \rightarrow 0,$$

where F and G have pure codimension p , and H has codimension $\geq p + 2$. If F is S_2 , then $F \cong G$.

We also obtain the following way of constructing new modules which are also S_2 .

Corollary 5.5. *Let G and F be finitely generated \mathcal{O} -modules, and assume that F is S_2 . Then $\text{Hom}(G, F)$ is S_2 .*

Proof. If \mathcal{G} is a coherent analytic sheaf, then $\mathcal{H}om(\mathcal{G}, \mathcal{F})_x \cong \text{Hom}(G_x, F_x)$, so it is enough to prove the corresponding statement for coherent analytic sheaves \mathcal{G} and \mathcal{F} and assume that \mathcal{F} is S_2 .

By Proposition 5.3, it is enough to prove that for any open set $U \subseteq X$, subvariety $V \subseteq U$ of codimension $\geq p + 2$ and $\alpha \in \mathcal{H}om(\mathcal{G}, \mathcal{F})(U \setminus V)$, α has a unique extension in $\mathcal{H}om(\mathcal{G}, \mathcal{F})(U)$. To prove this, consider such an α , and for some subset $W \subseteq U$, some $g \in \mathcal{G}(W)$. Then, $\alpha(g|_{W \setminus V}) \in \mathcal{F}(W \setminus V)$, and since \mathcal{F} is S_2 , this section has a unique extension to a section of $\mathcal{F}(W)$ by Proposition 5.3, and we will denote this extension $\tilde{\alpha}(g)$. Then, $\tilde{\alpha}$ is a unique extension to $\mathcal{H}om(\mathcal{G}, \mathcal{F})(U)$ of α . \square

Lemma 5.6. *The inclusion $G_{(p)} \rightarrow G$ induces a short exact sequence*

$$(5.9) \quad 0 \rightarrow \operatorname{Ext}^p(G, \mathcal{O})^{(p)} \rightarrow \operatorname{Ext}^p(G_{(p)}, \mathcal{O}) \rightarrow H \rightarrow 0,$$

where H has codimension $\geq p + 2$.

Proof. Consider the short exact sequence

$$0 \rightarrow G_{(p)} \rightarrow G \rightarrow H' \rightarrow 0,$$

where $H' := G/G_{(p)}$. From the long exact sequence of Ext , we have an exact sequence

$$\operatorname{Ext}^p(H', \mathcal{O}) \rightarrow \operatorname{Ext}^p(G, \mathcal{O}) \rightarrow \operatorname{Ext}^p(G_{(p)}, \mathcal{O}) \rightarrow \operatorname{Ext}^{p+1}(H', \mathcal{O}),$$

and since H' has only associated primes of codimension $< p$, $\operatorname{Ext}^p(H', \mathcal{O})$ has codimension $\geq p + 1$ by (5.6), so we get a short exact sequence (5.9), where $H := \operatorname{im}(\operatorname{Ext}^p(G_{(p)}, \mathcal{O}) \rightarrow \operatorname{Ext}^{p+1}(H', \mathcal{O}))$. By (5.6), $\operatorname{Ext}^{p+1}(H', \mathcal{O})$ has codimension $\geq p + 2$, since H' has only associated primes of codimension $< p$, and thus H , being a submodule of $\operatorname{Ext}^{p+1}(H', \mathcal{O})$, has codimension $\geq p + 2$. \square

5.2. Proof of Theorem 1.3. We can now give a full proof of the remaining parts of Theorem 1.3, including non-degeneracy of the pairing, (1.9) and surjectivity and non-surjectivity of the induced morphisms (1.10) and (1.11). Due to the length of the proof, and in order to structure it better, this proof is divided into several different steps. In some of the proofs, it is implicitly assumed that $p \geq 1$, since as is explained in the introduction, the case $p = 0$ of Theorem 1.3 is already well-known.

Proposition 5.7. *If the pairing (1.8) is given as $\langle \bullet, \bullet \rangle_{Gr}$, as defined in (2.9), then the induced pairing (1.9) is non-degenerate in the first argument.*

Proof. By the Nullstellensatz, we can find J such that $Z(J) \supseteq Z$ and $J \operatorname{Ext}^p(G, \mathcal{O}) = 0$. Take $I = J(f_1, \dots, f_p)$ a complete intersection ideal contained in J . By Lemma 4.5, it is enough to prove that

$$G^{(p)} \rightarrow \operatorname{Hom}(\operatorname{Ext}^p(G, \mathcal{O}), \operatorname{Ext}^p(\mathcal{O}/I, \mathcal{O}))$$

is injective, where the map is given by ϵ_g^* , where $\epsilon_g : \mathcal{O}/I \rightarrow G$, where $g \in G$ is a representative of an element in $G^{(p)}$.

Assume that $g \in G_{(p)}$, and $\epsilon_g^* = 0$. We then want to prove that $g \in G_{(p+1)}$. We let (K, ψ) be the Koszul complex of f , which is a free resolution of \mathcal{O}/I . Let $a : (K, \psi) \rightarrow (E, \varphi)$ be a morphism of complexes extending ϵ_g .

Note that by (5.4) and (5.5), $G^{(p)} \neq 0$ if and only if $\operatorname{codim} Z_p = p$. If $G^{(p)} = 0$, then non-degeneracy in the first argument is trivial, so we can assume that $\operatorname{codim} Z_p = p$. Let $h \in \mathcal{O}$ be a holomorphic function such that $\{h = 0\} \supseteq Z_{p+1}$ which does not vanish identically on any irreducible component of Z_p .

Since $\epsilon_g^* = 0$, $\alpha_p^*(\ker \varphi_{p+1}^*) \subseteq \operatorname{im} \psi_p^*$. By considering the long exact sequence of Ext , we get an exact sequence

$$\operatorname{Hom}(E_p^*, \operatorname{im} \psi_p^*) \rightarrow \operatorname{Hom}(\ker \varphi_{p+1}^*, \operatorname{im} \psi_p^*) \rightarrow \operatorname{Ext}^1(E_p^*/\ker \varphi_{p+1}^*, \operatorname{im} \psi_p^*).$$

Since $H := E_p^*/\ker \varphi_{p+1}^*$ is free outside of Z_{p+1} , $\operatorname{supp} \operatorname{Ext}^1(H, \operatorname{im} \psi_p^*) \subseteq Z_{p+1}$, and hence, by the Nullstellensatz, $h^N \operatorname{Ext}^1(H, \operatorname{im} \psi_p^*) = 0$ if $N \gg 1$.

Thus, we can extend $h^N a_p^*|_{\ker \varphi_{p+1}^*}$ to a map $\tilde{b}_p : E_p^* \rightarrow \text{im } \psi_p^*$. We let b_p be \tilde{b}_p composed with the inclusion $\text{im } \psi_p^* \rightarrow K_p^*$.

For $k < p$, we let $b_k := h^N a_k^*$. Since $b_p = h^N a_p^*$ on $\ker \varphi_{p+1}$, we get that $b_p \varphi_p^* = h^N a_p^* \varphi_p^* = h^N \psi_p^* a_{p-1}^* = \psi_p^* b_{p-1}$. In addition, $b_k \varphi_k^* = \psi_k^* b_{k-1}$ for $1 \leq k < p$, so $b : (E^*, \varphi^*) \rightarrow (K^*, \psi^*)$ is a morphism of complexes. Thus, $b^* : (E, \varphi) \rightarrow (K, \psi)$ is a morphism of complexes, and since $b_0^* = h^N a_0$, b^* extends $h^N \epsilon_g = \epsilon_{(h^N g)}$.

By definition of b_p , $\text{im } b_p \subseteq \text{im } \psi_p^*$, and since E_p^* is projective, we can lift b_p by ψ_p^* , i.e., there exist a morphism $s_p : E_p^* \rightarrow K_{p-1}^*$ such that $b_p = \psi_p^* s_p$. Now, $\psi_p^*(b_{p-1} - s_p \varphi_p^*) = 0$. Since $H^{p-1}(K_\bullet^*) \cong \text{Ext}^{p-1}(\mathcal{O}/I, \mathcal{O}) = 0$, $\text{im}(b_{p-1} - s_p \varphi_p^*) \subseteq \text{im } \psi_{p-1}^*$. Since E_{p-1}^* is projective, there exists $s_{p-1} : E_{p-1}^* \rightarrow K_{p-2}^*$ such that $b_{p-1} = s_p \varphi_p^* + \psi_{p-1}^* s_{p-1}$. Repeating this, we obtain $s_k : E_k^* \rightarrow K_{k-1}^*$ such that

$$b_k = s_{k+1} \varphi_{k+1}^* + \psi_k^* s_k.$$

In particular, $b_0 = s_1 \varphi_1^*$, so $h^N a_0 = b_0^* = \varphi_1 s_1^*$. Thus, $h^N \epsilon_g = 0$, so $h^N g = 0$, which implies that $\text{supp } g \subseteq \{h = 0\}$. By selecting h_1, \dots, h_ℓ such that h_i satisfies the conditions on h above, and such that $\{h_1 = \dots = h_\ell = 0\} = Z_{p+1}$, we see that $\text{supp } g \subseteq Z_{p+1}$, i.e., $g \in G_{(p+1)}$ since $\text{codim } Z_{p+1} \geq p+1$ by (5.4). \square

Regarding non-degeneracy in the second argument of (1.3), and surjectivity of the induced map (1.7), we first consider it in the case when $\text{codim } G \geq p$. The following is an immediate corollary of Corollary 5.1 and Proposition 5.2.

Corollary 5.8. *Let G be a finitely generated \mathcal{O} -module of codimension $\geq p$, and let $Z \subseteq (\mathbb{C}^n, 0)$ be a subvariety of pure codimension p such that $\text{supp } G \subseteq Z$. Then*

$$\text{Ext}^p(G, \mathcal{O}) \cong \text{Hom}(G, H_Z^p(\mathcal{O})).$$

We can then consider non-degeneracy in the second argument for general G .

Proposition 5.9. *If the pairing (1.8) is given as $\langle \bullet, \bullet \rangle_{Gr}$, as defined in (2.9), then the induced pairing (1.9) is non-degenerate in the second argument.*

Proof. By Lemma 5.6 and Proposition 2.3, we have an injection

$$0 \rightarrow \text{Ext}^p(G, \mathcal{O})^{(p)} \rightarrow \text{Ext}^p(G_{(p)}, \mathcal{O}) \cong \text{Ext}^p(G_{(p)}, \mathcal{O})^{(p)}.$$

It thus suffices to prove that (1.9) is non-degenerate in the second argument for $G = G_{(p)}$, which follows from Corollary 5.8. \square

We now turn to surjectiveness of the maps induced by the pairing (1.9). First, we consider the case of pure dimension.

Lemma 5.10. *Let G be a finitely generated \mathcal{O} -module of pure codimension p , and assume that G is S_2 . Then the map (1.10) is surjective.*

Proof. We choose I as in the proof of Proposition 5.7, and as in that proof, it is enough to prove that

$$G \rightarrow \operatorname{Hom}(\operatorname{Ext}^p(G, \mathcal{O}), \operatorname{Ext}^p(\mathcal{O}/I, \mathcal{O}))$$

is surjective, where the map is given by ϵ_g^* , where $\epsilon_g : \mathcal{O}/I \rightarrow G$.

We first show that the map is surjective outside of Z_{p+1} . We let (E, φ) and (K, ψ) be free resolutions of G and \mathcal{O}/I respectively, where we take (K, ψ) to be the Koszul complex of a tuple $f = (f_1, \dots, f_p)$ generating I . If we represent the Ext-groups with the help of these free resolutions, we can consider an element $\gamma' \in \operatorname{Hom}(\operatorname{Ext}^p(G, \mathcal{O}), \operatorname{Ext}^p(\mathcal{O}/I, \mathcal{O}))$ as a morphism $\ker \varphi_{p+1}^* \rightarrow K_p^*$. We claim that outside of Z_{p+1} , we can locally extend γ' to a morphism $\operatorname{Hom}(E_p^*, K_p^*)$. To see this, we consider the long exact sequence of Ext, which shows that obstruction to extending γ' lies in $E' := \operatorname{Ext}^1(E_p^*/\ker \varphi_{p+1}^*, K_p^*)$. Note that K_p^* is free, and $E_p^*/\ker \varphi_{p+1}^*$ is free outside of Z_{p+1} , so $\operatorname{supp} E' \subseteq Z_{p+1}$. Hence, outside of Z_{p+1} , we can locally extend γ' . We call this morphism γ_p .

Since γ_p equals γ' on $\operatorname{im} \varphi_p^* \subseteq \ker \varphi_{p+1}^*$, and γ' induces a map $H^p(E_\bullet^*) \rightarrow H^p(K_\bullet^*)$, $\gamma'_p \varphi_p^* \in \operatorname{im} \psi_p^*$. Thus, $\gamma_p \varphi_p^* \in \operatorname{im} \psi_p^*$, and similarly to the proof of Proposition 2.5, γ_p can be extended to a morphism of complexes $\gamma : (E^*, \varphi^*) \rightarrow (K^*, \psi^*)$. Let $a = \gamma^*$, which is a morphism of complexes $(K, \psi) \rightarrow (E, \varphi)$. Let $\alpha : \mathcal{O}/I \rightarrow G$ be the morphism induced by a_0 . If we let $g = \alpha(1) \in G$, then since $a^* = \gamma$, we get that

$$\epsilon_g^* : \operatorname{Ext}^p(G, \mathcal{O}) \cong H^p(\operatorname{Hom}(E_\bullet, \mathcal{O})) \rightarrow \operatorname{Ext}^p(\mathcal{O}/I, \mathcal{O}) \cong H^p(\operatorname{Hom}(K_\bullet, \mathcal{O}))$$

is given by γ' .

We have thus proven that (1.10) is surjective outside of Z_{p+1} , which has codimension $\geq p+2$ by (5.5), and since G is assumed to be S_2 , (1.10) is an isomorphism by Corollary 5.4. \square

Proposition 5.11. *Let G be a finitely generated \mathcal{O} -module, and assume that $G^{(p)}$ is S_2 . Then the map (1.10) is surjective.*

Proof. From the long exact sequence of Ext applied to the short exact sequence (5.9), we get the exact sequence

$$\begin{aligned} \operatorname{Hom}(H, H_Z^p(\mathcal{O})) &\rightarrow \operatorname{Hom}(\operatorname{Ext}^p(G_{(p)}, \mathcal{O}), H_Z^p(\mathcal{O})) \rightarrow \\ &\rightarrow \operatorname{Hom}(\operatorname{Ext}^p(G, \mathcal{O})^{(p)}, H_Z^p(\mathcal{O})) \rightarrow \operatorname{Ext}^1(H, H_Z^p(\mathcal{O})). \end{aligned}$$

Since H has codimension $\geq p+2$, and $H_Z^p(\mathcal{O})$ has pure codimension p by Proposition 2.3, $\operatorname{Hom}(H, H_Z^p(\mathcal{O})) = 0$. If we let

$$H' := \operatorname{im}(\operatorname{Hom}(\operatorname{Ext}^p(G, \mathcal{O})^{(p)}, H_Z^p(\mathcal{O})) \rightarrow \operatorname{Ext}^1(H, H_Z^p(\mathcal{O}))),$$

then $\operatorname{supp} H' \subseteq \operatorname{supp} H$, which has codimension $\geq p+2$, so we get a short exact sequence

$$0 \rightarrow \operatorname{Hom}(\operatorname{Ext}^p(G_{(p)}, \mathcal{O}), H_Z^p(\mathcal{O})) \rightarrow \operatorname{Hom}(\operatorname{Ext}^p(G, \mathcal{O})^{(p)}, H_Z^p(\mathcal{O})) \rightarrow H' \rightarrow 0,$$

where H' has codimension $\geq p+2$. From the long exact sequence of Ext from the short exact sequence

$$0 \rightarrow G_{(p+1)} \rightarrow G_{(p)} \rightarrow G^{(p)} \rightarrow 0,$$

one gets an isomorphism $\text{Ext}^p(G^{(p)}, \mathcal{O}) \cong \text{Ext}^p(G_{(p)}, \mathcal{O})$. Thus, we have a short exact sequence

$$(5.10) \quad 0 \rightarrow \text{Hom}(\text{Ext}^p(G^{(p)}, \mathcal{O}), H_Z^p(\mathcal{O})) \rightarrow \text{Hom}(\text{Ext}^p(G, \mathcal{O})^{(p)}, H_Z^p(\mathcal{O})) \rightarrow H' \rightarrow 0.$$

By Lemma 5.10, we have an isomorphism

$$G^{(p)} \cong \text{Hom}(\text{Ext}^p(G^{(p)}, \mathcal{O}), H_Z^p(\mathcal{O})),$$

and inserting this in (5.10), we get an exact sequence

$$0 \rightarrow G^{(p)} \rightarrow \text{Hom}(\text{Ext}^p(G, \mathcal{O})^{(p)}, H_Z^p(\mathcal{O})) \rightarrow H' \rightarrow 0,$$

which is an isomorphism by Corollary 5.4. \square

Proposition 5.12. *Let G be a finitely generated \mathcal{O} -module, and assume that $\text{Ext}^p(G, \mathcal{O})^{(p)}$ is S_2 . Then the map (1.11) is surjective.*

Proof. By Corollary 5.4 applied to the short exact sequence (5.9),

$$\text{Ext}^p(G, \mathcal{O})^{(p)} \rightarrow \text{Ext}^p(G_{(p)}, \mathcal{O})$$

is an isomorphism since $\text{Ext}^p(G, \mathcal{O})^{(p)}$ is assumed to be S_2 . By Corollary 5.8, applied with $G_{(p)}$,

$$\text{Ext}^p(G_{(p)}, \mathcal{O}) \cong \text{Hom}(G_{(p)}, H_Z^p(\mathcal{O})),$$

and finally, since $H_Z^p(\mathcal{O})$ has pure codimension p by Proposition 2.3, the surjection $G_{(p)} \rightarrow G^{(p)}$ induces an isomorphism

$$\text{Hom}(G^{(p)}, H_Z^p(\mathcal{O})) \cong \text{Hom}(G_{(p)}, H_Z^p(\mathcal{O})).$$

\square

Proposition 5.13. *Let G be a finitely generated \mathcal{O} -module. If $G^{(p)}$ is not S_2 , then (1.10) is not surjective, and if $\text{Ext}^p(G, \mathcal{O})^{(p)}$ is not S_2 , then the map (1.11) is not surjective.*

Proof. By Proposition 5.7 and Proposition 5.9, (1.10) and (1.11) are injective, and thus isomorphisms if and only if they are surjective. We are thus finished if we show that $\text{Hom}(\text{Ext}^p(G, \mathcal{O})^{(p)}, H_Z^p(\mathcal{O}))$ and $\text{Hom}(G^{(p)}, H_Z^p(\mathcal{O}))$ are always S_2 . If we take a complete intersection ideal $I \subseteq \text{ann } G^{(p)}$ and $I \subseteq \text{ann } \text{Ext}^p(G, \mathcal{O})^{(p)}$ of codimension p , then by Lemma 4.5,

$$(5.11) \quad \text{Hom}(\text{Ext}^p(G, \mathcal{O})^{(p)}, H_Z^p(\mathcal{O})) \cong \text{Hom}(\text{Ext}^p(G, \mathcal{O})^{(p)}, \text{Ext}^p(\mathcal{O}/I, \mathcal{O}))$$

and

$$(5.12) \quad \text{Hom}(G^{(p)}, H_Z^p(\mathcal{O})) \cong \text{Hom}(G^{(p)}, \text{Ext}^p(\mathcal{O}/I, \mathcal{O})).$$

Since $\text{Ext}^p(\mathcal{O}/I, \mathcal{O}) \cong \mathcal{O}/I$, and \mathcal{O}/I is S_2 , we get that both (5.11) and (5.12) are S_2 by Corollary 5.5. \square

Remark 5.14. As mentioned after Theorem 1.3, it is implied by our results that $\text{Ext}^p(G, \mathcal{O})^{(p)}$ is S_2 if G has codimension $\geq p$. Here, we explain how this can also be seen more directly using a result of Björk.

We assume that $\text{Ext}^p(G, \mathcal{O}) \neq 0$. By Proposition 2.3, $\text{Ext}^p(G, \mathcal{O})$ has pure codimension p . Then, we want to show that $\text{codim } Z_\ell(\text{Ext}^p(G, \mathcal{O})) \geq \ell + 2$

for $\ell > p$. By for example the proof of [BS, Theorem II.2.1], for a finitely generated \mathcal{O} -module H ,

$$Z_\ell(H) = \cup_{r \geq \ell} \text{supp Ext}^r(H, \mathcal{O}).$$

It is thus enough to prove that $\text{Ext}^r(\text{Ext}^p(G, \mathcal{O}), \mathcal{O})$ has codimension $\geq r+2$ for $r > p$. This is the case, since by [B1, Lemma 2.7.11], if H has codimension p , then

$$\text{codim Ext}^k(\text{Ext}^p(H, \mathcal{O}), \mathcal{O}) \geq k+2$$

for $k > p$.

Applying this for $H = \text{Ext}^p(G, \mathcal{O})$, which has codimension $\geq p$, we get that $\text{Ext}^p(\text{Ext}^p(G, \mathcal{O}), \mathcal{O})^{(p)}$ is S_2 , and since $\text{Ext}^p(\text{Ext}^p(G, \mathcal{O}), \mathcal{O})$ has pure codimension p by Proposition 2.3, it is S_2 .

6. THE ANDERSSON-WULCAN PAIRING

In this section, we give direct analytic expressions for the pairings (1.2) and (1.8) with the help of residue currents, when $H_Z^p(\mathcal{O})$ is represented as currents as in (3.6).

6.1. Coleff-Herrera currents. By (3.6), we can represent elements in $H_Z^p(\mathcal{O})$ as $\bar{\partial}$ -closed $(0, p)$ -currents with support in Z modulo $\bar{\partial}$ of such $(0, p-1)$ -currents. When discussing cohomological residues below, it will be useful that there is in fact a canonical choice of representative in each such cohomology class. In fact, we also get just as in [A2] that our second way of defining the pairing indeed gives directly this representative, see Remark 6.11.

So called Coleff-Herrera currents were introduced in [DS1] (under the name “locally residual currents”), as canonical representatives of certain local cohomology classes. Let $(Z, 0)$ be the germ of a subvariety of $(\mathbb{C}^n, 0)$ of pure codimension p . A $(*, p)$ -current μ on $(\mathbb{C}^n, 0)$ is a *Coleff-Herrera current*, denoted $\mu \in CH_Z$, if $\bar{\partial}\mu = 0$, $\bar{\psi}\mu = 0$ for all holomorphic functions ψ vanishing on Z , and μ has the *standard extension property*, SEP, with respect to Z . We say that μ has the SEP if the limit $\lim_{\epsilon \rightarrow 0^+} \chi(|h|^2/\epsilon)\mu$ exists and is 0, for any tuple of holomorphic functions h such that $\{h=0\} \cap Z$ has codimension $> p$, where $\chi(t) : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth cut-off function which is identically 0 for t close to 0 and which is identically 1 for $t \geq 1$.

This description of Coleff-Herrera currents is due to Björk, see [B2], Chapter 3, and [B3], Section 6.2. In [DS1], locally residual currents on Z were defined as currents of the form $\mu = \omega \wedge R$, with support on Z , where ω is a holomorphic $(*, 0)$ -form, and R is a Coleff-Herrera product of a tuple (f_1, \dots, f_p) defining a complete intersection ideal $I = J(f_1, \dots, f_p)$ of codimension p .

As mentioned above, one of the main objectives of [DS1], and later refined in [DS2], was to obtain canonical representatives of local cohomology classes in $H_Z^p(\mathcal{O})$ in terms of Coleff-Herrera products. By (3.6), $H_Z^p(\mathcal{O})$ is canonically isomorphic to $H^p(C_Z^{0, \bullet})$. By Theorem 5.1 in [DS2],

$$\ker(\mathcal{C}_Z^{0, p} \xrightarrow{\bar{\partial}} \mathcal{C}_Z^{0, p+1}) = CH_Z \oplus \bar{\partial}C_Z^{0, p-1},$$

and one thus obtains an isomorphism

$$(6.1) \quad CH_Z \xrightarrow{\cong} H^p(C^{0,\bullet}) \cong H_Z^p(\mathcal{O}),$$

so each element in $H_Z^p(\mathcal{O})$ has a unique representative as a current $\mu \in CH_Z$.

6.2. Residue currents of Andersson-Wulcan. The duality theorem for Coleff-Herrera products, (3.5), was generalized by Andersson and Wulcan in [AW1]. Let (E, φ) be a free resolution of length N of a finitely generated \mathcal{O} -module G of codimension $p > 0$, such that $\varphi_1 : E_1 \rightarrow E_0$ is generically surjective, and assume that E_0, \dots, E_N are equipped with Hermitian metrics. Andersson and Wulcan constructed in [AW1] an associated $\text{Hom}(E_0, E)$ -valued residue current R^E in satisfying the following duality principle [AW1, Theorem 1.1]: if $g_0 \in E_0$, then

$$(6.2) \quad R^E g_0 = 0 \text{ if and only if } g_0 \in \text{im } \varphi_1.$$

The current R^E can be decomposed in the form

$$(6.3) \quad R^E = \sum_{k=p}^N R_k^E.$$

where R_k^E is a $\text{Hom}(E_0, E_k)$ -valued $(0, k)$ -current. These currents satisfy that

$$(6.4) \quad \varphi_k R_k^E = \bar{\partial} R_{k-1}^E,$$

see [AW1, Proposition 2.2].

Let $G = \mathcal{O}/I$, where $I = (f_1, \dots, f_p)$ is a complete intersection ideal of codimension p , and let (E, φ) be the Koszul complex of (f_1, \dots, f_p) . Then it is not only the case that the Coleff-Herrera product of f and R^E have the same annihilator, i.e., I , but they do in fact coincide. More precisely, if we let e_1, \dots, e_p be a trivial frame of $K_1 = \mathcal{O}^p$ such that the differential in the Koszul complex is contraction with $\sum f_i e_i^*$, and $e_1 \wedge \dots \wedge e_p$ is the induced frame on K_p , then

$$(6.5) \quad R^E = R_p^E = \bar{\partial} \frac{1}{f_p} \wedge \dots \wedge \bar{\partial} \frac{1}{f_1} \wedge e_1 \wedge \dots \wedge e_p,$$

see [A1, Corollary 3.2] and [PTY, Theorem 4.1]. Thus, the duality principle of Andersson-Wulcan is a direct generalization of the duality theorem for Coleff-Herrera products.

As introduced in [AW2], a current of the form

$$\frac{1}{z_{i_1}^{n_1}} \dots \frac{1}{z_{i_k}^{n_k}} \bar{\partial} \frac{1}{z_{i_{k+1}}^{n_{k+1}}} \wedge \dots \wedge \bar{\partial} \frac{1}{z_{i_m}^{n_m}} \wedge \omega,$$

in some local coordinate system z , where ω is a smooth form with compact support is said to be an *elementary current*, and a current on a complex manifold is said to be *pseudomeromorphic*, if it can be written as a locally finite sum of push-forwards of elementary currents under compositions of modifications and open inclusions. As can be seen from the construction, the Andersson-Wulcan currents R_k^E are pseudomeromorphic.

An important property of pseudomeromorphic currents is that they satisfy the following *dimension principle*, [AW2, Corollary 2.4].

Proposition 6.1. *If T is a pseudomeromorphic $(*, q)$ -current with support on a variety Z , and $\text{codim } Z > q$, then $T = 0$.*

This is a variant for pseudomeromorphic currents of the SEP, which we described above for currents in CH_Z .

6.3. A comparison formula for residue currents. The following is a generalization of the transformation law for Coleff-Herrera products to Andersson-Wulcan currents, and which is expressed with the help of the comparison morphism as in Proposition 2.5. We will use the following somewhat simplified version of [L3, Theorem 3.2]. The last part in the statement of the theorem is part of [L3, Corollary 3.6].

Theorem 6.2. *Let F and G be finitely generated \mathcal{O} -modules and let (E, φ) and (K, ψ) be free resolutions of G and F . Let $\alpha : F \rightarrow G$ be a morphism, and let $a : (K, \psi) \rightarrow (E, \varphi)$ be a morphism of complexes, extending α as in Proposition 2.5. Then, there exist pseudomeromorphic $\text{Hom}(K_0, E_k)$ -valued $(0, k-1)$ -currents M_k such that*

$$(6.6) \quad R_p^E a_0 - a_p R_p^K = \varphi_{p+1} M_{p+1} - \bar{\partial} M_p.$$

For any F and G , $M_p \psi_1 = 0$, and if F and G have codimension $\geq p$, then $M_p = 0$.

One basic special case of Theorem 6.2 is when $F = \mathcal{O}/I$ and $G = \mathcal{O}/J$, where $I = J(f_1, \dots, f_p)$ and $J = J(g_1, \dots, g_p)$ are complete intersection ideals of codimension p , and $I \subseteq J$, or equivalently, there exist a matrix holomorphic $(p \times p)$ -matrix A such that $(f_1, \dots, f_p) = (g_1, \dots, g_p)A$. By Example 3.4, if we take (E, φ) and (K, ψ) to be the Koszul complexes of g and f respectively, then $a_p = \det A$. In addition, by (6.5), R^E and R^K are the Coleff-Herrera products of g and f respectively. In addition, since $E_{p+1} = 0$, $M_{p+1} = 0$, since it is $\text{Hom}(E_0, E_{p+1})$ -valued, and since $Z(I)$ and $Z(J)$ have codimension p , $M_p = 0$. Thus, in this case, the comparison formula, (6.6), becomes the transformation law for Coleff-Herrera products, (3.11).

6.4. Definitions and properties of the pairing. We here give an alternative expressions for our pairing. This pairing is defined with the help of residue current of Andersson and Wulcan from [AW1]. The pairing appears explicitly in [A2], in the case when G has pure codimension p , including a proof that it then is non-degenerate, but it is not proven there that the pairing is functorial in G .

If (E, φ) is a free resolution of G , then the associated residue current R_p^E as in (6.3) takes values in $\text{Hom}(E_0, E_p)$. We use the identification of $\text{Ext}^p(G, \mathcal{O})$ with $H^p(\text{Hom}(E_\bullet, \mathcal{O}))$, and one can thus represent an element $\xi \in \text{Ext}^p(G, \mathcal{O})$ as

$$(6.7) \quad \xi = [\xi_0], \text{ where } \xi_0 \in \text{Hom}(E_p, \mathcal{O}) \text{ is such that } \varphi_{p+1}^* \xi_0 = 0.$$

In addition, for $g \in G_{(p)}$, we choose a representative

$$(6.8) \quad g_0 \in E_0 \text{ such that } g = \pi(g_0) \text{ where } \pi : E_0 \rightarrow \text{coker } \varphi_1 \cong G$$

is the natural surjection.

The *Andersson-Wulcan pairing* is defined by

$$(6.9) \quad \langle g, \xi \rangle_{AW} := \xi_0 R_p^E g_0,$$

where ξ_0 and g_0 are as in (6.7) and (6.8). Here, $\xi_0 R_p^E g_0$ is considered as an element of $H_Z^p(\mathcal{O})$ using the representation (3.6), i.e., we claim that $\xi_0 R_p^E g_0$ is a $\bar{\partial}$ -closed current in $C_Z^{0,p}$. The fact that it is a $\bar{\partial}$ -closed $(0, p)$ -current follows exactly as in [A2]. That it has its support on Z follows from that outside of $\text{supp } g \subseteq \text{supp } G_{(p)} \subseteq Z$, $g_0 \in \text{im } \varphi_1$, and hence, $\text{supp } \xi_0 R_p^E g_0 \subseteq \text{supp } g$ by (6.2).

Lemma 6.3. *The Andersson-Wulcan pairing (6.9) is well-defined, and functorial in G . More precisely, if $\alpha : F \rightarrow G$ is a morphism, (K, ψ) and (E, φ) are free resolutions of F and G respectively, and $a : (K, \psi) \rightarrow (E, \varphi)$ is a morphism of complexes extending α , and $a_0 f_0 = g_0$, where $f_0 \in K_0$ and $g_0 \in E_0$ are representatives of $f \in F_{(p)}$ and $g \in G_{(p)}$, then*

$$(6.10) \quad \xi_0 R_p^E g_0 = a_p^* \xi_0 R_p^K f_0$$

as currents.

Note that in order to prove functoriality, it would be enough to prove that (6.10) holds in (3.6), i.e., as cohomology classes, so that (6.10) holds modulo $\bar{\partial}$ of a current vanishing on Z .

Proof. The fact that (6.9) is independent of the choice of representative g_0 follows just as in [A2, Theorem 1.2]. That it is independent of the choice of ξ_0 is proven as follows. If ξ_1 is another representative of $[\xi_0]$, then ξ_0 and ξ_1 differ by something in $\text{im } \varphi_p^*$. Thus, it suffices to prove that

$$\varphi_p R_p^E g_0 = 0.$$

By (6.4), $\varphi_p R_p^E = \bar{\partial} R_{p-1}^E$, so using this, and the fact that g_0 is $\bar{\partial}$ -closed,

$$\varphi_p R_p^E g_0 = \bar{\partial}(R_{p-1}^E g_0).$$

Since $g \in G_{(p)}$, $g = 0$ outside of $\text{supp } G_{(p)}$ which has codimension $\geq p$, so $\text{supp } R_{p-1}^E g_0$ is contained in a variety of codimension $\geq p$, and since it is a pseudomeromorphic $(0, p-1)$ -current, it is 0 by the dimension principle, Proposition 6.1, so $\varphi_p R_p^E g_0 = 0$.

Regarding independence of the choice of (E, φ) with Hermitian metrics on E , it is a special case of functoriality in (6.10), which we prove below, when we take $a : (K, \psi) \rightarrow (E, \varphi)$ to be the morphism induced by the identity map on G , if (K, ψ) and (E, φ) are both free resolutions of G .

We now prove (6.10). Note that if $f \in F$ and $f_0 \in K_0$ satisfies $\pi(f_0) = f$ (cf., (6.8)), then $\pi(a_0 f_0) = \alpha f$. Thus, by the comparison formula, (6.6),

$$\begin{aligned} \langle \alpha f, \xi \rangle_{AW} &= \xi_0 R_p^E a_0 f_0 = \xi_0 a_p R_p^K f_0 + \xi_0 \varphi_{p+1} M_{p+1} f_0 - \bar{\partial}(\xi_0 M_p f_0) = \\ &= \langle f, \alpha^* \xi \rangle_{AW} + \xi_0 \varphi_{p+1} M_{p+1} f_0 - \bar{\partial}(\xi_0 M_p f_0), \end{aligned}$$

and it remains to prove that the last two terms vanish. We get that $\xi_0 \varphi_{p+1} M_{p+1} f_0 = 0$ since $\varphi_{p+1}^* \xi_0 = 0$. In addition, outside of $\text{supp } F_{(p)}$, $f_0 = \psi_1 f_1$ for some f_1 , so $\text{supp } M_p f_0 \subseteq \text{supp } F_{(p)}$, which has codimension p . Thus, $M_p f_0 = 0$, since its support has codimension $\geq p$, and it is a

pseudomeromorphic $(0, p-1)$ -current, so it is 0 by the dimension principle, Proposition 6.1. \square

Example 6.4. We now consider the most basic case when $G = \mathcal{O}/I$, where I is a complete intersection ideal of codimension p , given as $I = J(f_1, \dots, f_p)$. Then, the Koszul complex (K, ψ) of f is a free resolution of \mathcal{O}/I . Using the representation (3.3) of $\text{Ext}^p(\mathcal{O}/I, \mathcal{O})$ together with the expression (6.5) for the current R_p^K , we get that the pairing (6.9) in this case becomes

$$\langle g, h(e_1 \wedge \dots \wedge e_p)^* \rangle_{AW} = gh \bar{\partial} \frac{1}{f_p} \wedge \dots \wedge \bar{\partial} \frac{1}{f_1}.$$

In this case, by (3.10), the pairing (2.9) coincides indeed with the pairing (6.9) using the canonical isomorphism (3.3) between the different representations of $\text{Ext}^p(\mathcal{O}/I, \mathcal{O})$ being used to define the pairings.

Proposition 6.5. *Let G be a finitely generated \mathcal{O} -module, and let $Z \subseteq (\mathbb{C}^n, 0)$ be a subvariety of pure codimension p . Under the canonical isomorphism between the representations of $H_Z^p(\mathcal{O})$ induced by the isomorphism (3.7), the pairings (2.9) and (6.9) coincide.*

Proof. It follows from Examples 6.4 that this holds when $G = \mathcal{O}/I$, where I is a complete intersection ideal of codimension p . By functoriality, and taking the morphism $\alpha : (\mathcal{O}/I)^r \rightarrow G$ from Lemma 4.1, which is surjective onto $G_{(p)}$, it follows from functoriality of both pairings that they coincide also in general.

Alternatively, one can prove the proposition in the following way. In [A2, Theorem 1.5], Andersson proves the following generalization of (3.8), that for G a finitely generated \mathcal{O} -module of pure codimension p and (E, φ) a free resolution of G , the canonical isomorphism

$$(6.11) \quad H^p(\text{Hom}(E_\bullet, \mathcal{O})) \cong H^p(\text{Hom}(G, C^{0,\bullet}))$$

is given by

$$(6.12) \quad \xi = [\xi_0] \mapsto \xi_0 R_p^E.$$

Here, R_p^E is considered as $\text{Hom}(G, C^{0,\bullet})$ -valued by $R_p^E(g) := R_p^E g_0$, where g_0 is a representative of $g \in G$, as in (6.8). We now let (K, ψ) be a free resolution of \mathcal{O}/J , and consider the following commutative diagram coming from the morphism $\epsilon_g : \mathcal{O}/J \rightarrow G$, which induces morphisms $\epsilon_g^* : \text{Ext}^p(G, \mathcal{O}) \rightarrow \text{Ext}^p(\mathcal{O}/J, \mathcal{O})$, and the canonical isomorphisms (3.8) and (6.11) between the different representations of Ext .

$$\begin{array}{ccc} H^p(\text{Hom}(E_\bullet, \mathcal{O})) & \xrightarrow{\cong} & H^p(\text{Hom}(G, C^{0,\bullet})) \\ \downarrow \epsilon_g^* & & \downarrow \epsilon_g^* \\ H^p(\text{Hom}(K_\bullet, \mathcal{O})) & \xrightarrow{\cong} & H^p(\text{Hom}(\mathcal{O}/J, C^{0,\bullet})). \end{array}$$

The map ϵ_g^* on the right is just precomposition with the map ϵ_g , so considering the element $\xi = [\xi_0] \in H^p(\text{Hom}(E_\bullet, \mathcal{O}))$, this will be mapped in the

diagram as follows:

$$\begin{array}{ccc} [\xi_0] & \longrightarrow & [\xi_0 R_p^E] \\ \downarrow & & \downarrow \\ \epsilon_g^*[\xi_0] & \longrightarrow & [\xi_0 R_p^E g_0]. \end{array}$$

Finally, using the representation (3.6), the map $\pi_1 : \text{Hom}(\mathcal{O}/K, C^{0,\bullet}) \rightarrow H_Z^p(\mathcal{O})$ acts just as the identity on currents, i.e.,

$$\pi_1[\xi_0 R_p^E g_0] = [\xi_0 R_p^E g_0] = \langle \xi, g \rangle_{AW} \in H_Z^p(\mathcal{O}).$$

□

Example 6.6. As we do in Section 4, and as well in the proof of the preceding lemma, we can reduce the expression of the pairing to the complete intersection case, which when expressed in terms of currents thus becomes an expression in terms of Coleff-Herrera products. We take the map $\alpha : (\mathcal{O}/I)^r \rightarrow G$ as in Lemma 4.1, and for $g \in G_{(p)}$, we take $h \in (\mathcal{O}/I)^r$ such that $\alpha(h) = g$, which is possible since α is surjective onto $G_{(p)}$. Then, by functoriality of the pairing and Example 6.4,

$$(6.13) \quad \langle g, \xi \rangle_{AW} = \bar{\partial} \frac{1}{f_p} \wedge \cdots \wedge \bar{\partial} \frac{1}{f_1} \wedge \xi_0 a_p(h e_1 \wedge \cdots \wedge e_p),$$

where ξ_0 is as in (6.7).

Example 6.7. In the special case above, when G is *Artinian*, i.e., when $\text{supp } G = \{0\}$, then by the Nullstellensatz, one can always choose the complete intersection ideal I in Lemma 4.1 to be of the form $I = J(z_1^{N_1}, \dots, z_n^{N_n})$, and one thus gets a representation

$$(6.14) \quad \langle g, \xi \rangle_{AW} = \bar{\partial} \frac{1}{z_1^{N_1}} \wedge \cdots \wedge \bar{\partial} \frac{1}{z_n^{N_n}} \wedge \xi_0 a_n(h e_1 \wedge \cdots \wedge e_n).$$

Remark 6.8. The equality (6.13) was one of the starting points of this article. If we in particular consider the Artinian case as in Example 6.7, then the existence of the pairing (6.14) is elementary, using only the Nullstellensatz and the syzygy theorem, while the existence of the currents in (6.9) is rather non-elementary also in this special situation (without using the comparison formula). One of the starting aims in writing this article was then to try to find also a more elementary proof of the non-degeneracy of the pairing defined by the right-hand side of (6.14), which is then achieved by the elementary proofs of non-degeneracy in the case when $f = (z_1^{N_1}, \dots, z_n^{N_n})$, combined with Lemma 4.4, which reduces non-degeneracy in the second argument to this case, and Lemma 4.8, which with the help of the theory of linkage reduces non-degeneracy in the first argument to this case.

Proposition 6.9. *If the pairing (1.8) is given as $\langle \bullet, \bullet \rangle_{AW}$, as defined in (6.9), then the induced pairing (1.9) is non-degenerate.*

Proof. For $G = \mathcal{O}/I$, where I is a complete intersection ideal, non-degeneracy in both arguments follows from the duality theorem for Coleff-Herrera products, as explained in Example 6.4. Thus, by Lemma 4.4, the pairing is non-degenerate in the second argument.

To prove non-degeneracy in the first argument, we assume first that we are outside of Z_{p+1} . Then G has a free resolution of length $\leq p$. Since the pairing is independent of the free resolution, we can thus assume that the free resolution (E, φ) of G has length $\leq p$. If $g \in G_{(p)}$, then $g \in G$ outside of $\text{supp } G_{(p)}$, so $R_k^E g_0 = 0$ outside of $\text{supp } G_{(p)}$ which has codimension $\geq p$. Hence, $R_k^E g_0 = 0$ for $k < p$ since it is a pseudomeromorphic $(0, k)$ -current with support on a subvariety of codimension $\geq p$, and thus is 0 by the dimension principle, Proposition 6.1. In addition, if (E, φ) has length $\leq p$, then $\ker \varphi_{p+1}^* = E_p^*$, so if $\xi R_p^E g_0 = 0$ for all $\xi \in \ker \varphi_{p+1}^*$, then $R_p^E g_0 = 0$. Thus, $R^E g_0 = 0$, so $g = 0$ by (6.2). To conclude, $g = 0$ outside of Z_{p+1} which has codimension $\geq p + 1$, by (5.4), so $g \in G_{(p+1)}$. \square

Example 6.10. The explicitness of the pairing (6.14) depends on the ability of calculating the morphism a_n . In [LW], we show that if J is an *Artinian monomial ideal*, i.e., an Artinian ideal generated by monomials, then one can explicitly compute the morphism a when (E, φ) is the so-called Hull resolution of \mathcal{O}/J .

In [LW], we then use the explicit expression of a to express the current R_n^E , which thus also give an explicit description of the pairing (6.14). For example, when $J \subseteq \mathcal{O}_{\mathbb{C}^2, w, 0}$ is the ideal $J = J(z^a, z^b w^c, w^d)$, where $b < a$ and $c < d$, then \mathcal{O}/J has the Hull resolution

$$0 \rightarrow \mathcal{O}^2 \xrightarrow{\varphi_2} \mathcal{O}^3 \xrightarrow{\varphi_1} \mathcal{O} \rightarrow \mathcal{O}/J \rightarrow 0,$$

where

$$\varphi_2 = \begin{bmatrix} -w^c & 0 \\ z^{a-b} & -w^{d-c} \\ 0 & z^b \end{bmatrix} \text{ and } \varphi_1 = \begin{bmatrix} z^a & z^b w^c & w^d \end{bmatrix},$$

and the current R_2^E is

$$R_2^E = \begin{bmatrix} \bar{\partial} \frac{1}{w^c} \wedge \bar{\partial} \frac{1}{z^a} \\ \bar{\partial} \frac{1}{w^d} \wedge \bar{\partial} \frac{1}{z^b} \end{bmatrix}.$$

If $\xi = [(\xi_1, \xi_2)^*] \in (\mathcal{O}^2)^*/(\text{im } \varphi_2^*) \cong \text{Ext}^2(\mathcal{O}/J, \mathcal{O})$, then the pairing (6.14) is

$$\langle g, \xi \rangle_{AW} = \xi R_2^E g = \xi_1 \bar{\partial} \frac{1}{w^c} \wedge \bar{\partial} \frac{1}{z^a} g + \xi_2 \bar{\partial} \frac{1}{w^d} \wedge \bar{\partial} \frac{1}{z^b} g.$$

The non-degeneracy of the pairing in the first argument thus corresponds to the decomposition of J ,

$$J = \text{ann} \left(\bar{\partial} \frac{1}{w^c} \wedge \bar{\partial} \frac{1}{z^a} \right) \cap \text{ann} \left(\bar{\partial} \frac{1}{w^d} \wedge \bar{\partial} \frac{1}{z^b} \right) = J(z^a, w^c) \cap J(z^b, w^d).$$

Remark 6.11. We finally also notice the difference in formulation of Theorem 1.1 and the main theorem in [A2]. By (6.1), we could just as well formulate (1.3) as that there exists a non-degenerate pairing

$$(6.15) \quad G/G_{(p+1)} \times \text{Ext}^p(G, \mathcal{O}) \rightarrow CH_Z,$$

which is indeed the formulation used in [A2]. Note that just as in [A2], if we use (6.9) to define the pairing, then this gives directly the representative in CH_Z .

Similarly, Theorem 1.2 and Theorem 1.3 can be reformulated as that there exists a canonical non-degenerate pairing

$$(6.16) \quad G_{(p)} \times \text{Ext}^p(G, \mathcal{O}) \rightarrow CH_Z,$$

such that the induced injective morphisms

$$(6.17) \quad G^{(p)} \rightarrow \text{Hom}(\text{Ext}^p(G, \mathcal{O})^{(p)}, CH_Z)$$

and

$$(6.18) \quad \text{Ext}^p(G, \mathcal{O})^{(p)} \rightarrow \text{Hom}(G^{(p)}, CH_Z)$$

are surjective if and only if $G^{(p)}$ and $\text{Ext}^p(G, \mathcal{O})^{(p)}$ are S_2 respectively, and if G has codimension $\geq p$, the latter is automatic, and one has an isomorphism

$$(6.19) \quad \text{Ext}^p(G, \mathcal{O}) \cong \text{Hom}(G^{(p)}, CH_Z).$$

7. COHOMOLOGICAL RESIDUES

In this section, we prove Theorem 1.6, and describe the alternative description of the pairing (1.22) when G has codimension $\geq p$.

7.1. Cohomological residues of Lundqvist. The description of the pairing (1.22) when G has codimension $\geq p$ is based on a construction by Lundqvist in [L1] and [L2]. In these articles, only the case when G is of the form $G = \mathcal{O}/J$, where J has pure codimension p is considered, but the construction of the pairing works the same also in this more general setting.

Let G be a finitely generated \mathcal{O} -module with a free resolution (E, φ) of length N , equipped with some Hermitian metrics, and let $Z = Z(\text{supp } G)$. We let Ω be a neighbourhood of 0 such that the free resolution exists and is pointwise exact on $\Omega \setminus Z$. By [AW1] (cf., [L2, Section 2]), there exist smooth $\text{Hom}(E_0, E_k)$ -valued $(0, k-1)$ -forms u_k for $k = 1, \dots, N$ and $\text{Hom}(E_1, E_k)$ -valued $(0, k-2)$ -forms u_k^1 for $k = 2, \dots, N$ on $\Omega \setminus Z$ such that

$$(7.1) \quad \begin{aligned} \varphi_1 u_1 &= \text{Id}_{E_0}, \quad u_1 \varphi_1 = \text{Id}_{E_1}, \quad \bar{\partial} u_N = 0, \\ \varphi_2 u_2^1 &= u_1 \varphi_1 + \text{Id}_{E_1} \quad \text{and} \quad \varphi_{k+1} u_{k+1}^1 = u_k \varphi_1 + \bar{\partial} u_k^1 \quad \text{for } k = 2, \dots, N. \end{aligned}$$

In [L2], this is expressed in the more compact notation $\nabla_{\text{End}(E)} u = \text{Id}_E$.

The main theorem in [L2], Theorem 3.3, can be reformulated as that if G is of the form $G = \mathcal{O}/J$, where J has pure codimension p , then

$$(7.2) \quad g \in \mathcal{O}/J \text{ if and only if } \int \xi_0 u_p g \wedge \bar{\partial} \beta = 0,$$

for all $\beta \in H_{Z^c}^{n, n-p}$ and $[\xi_0] \in H^p(\text{Hom}(E_\bullet, \mathcal{O})) \cong \text{Ext}^p(\mathcal{O}/J, \mathcal{O})$. In the case when $G = \mathcal{O}/I$, where I is a complete intersection ideal of codimension p , then u_p coincides with the form B_f defined by Passare, as in the introduction, and thus, the result of Lundqvist is a generalization of the result of Passare.

For future reference, we remark that the current R_p^E in Section 6.2 is defined as

$$(7.3) \quad R_p^E = \varphi_{p+1} U_{p+1} - \bar{\partial} U_p,$$

where U_p and U_{p+1} are currents on Ω which are the so-called standard extensions of u_p and u_{p+1} , which in particular means that u_p and u_{p+1} coincide with U_p and U_{p+1} where they are smooth.

7.2. A definition of the pairing (1.22). Even though the main result is not formulated in this way, the pairing still appears in the construction of Lundqvist, see [L2, (2) and (9)]. Let G be an finitely generated \mathcal{O} -module of codimension $\geq p$, and let $Z \supseteq \text{supp } G$ be of pure codimension p . The *Lundqvist pairing*

$$G \times \text{Ext}^p(G, \mathcal{O}) \rightarrow \text{Hom}_{\mathbb{C}}(H_{Z^c}^{n, n-p}, \mathbb{C})$$

is defined by

$$(7.4) \quad \langle g, \xi \rangle_{Lu}(\beta) := \int \xi_0 u_p g_0 \wedge \bar{\partial} \beta,$$

where ξ_0 and g_0 are as in (6.7) and (6.8), and $\beta \in H_{Z^c}^{n, n-p}$. It is implicitly assumed that β has small enough support such that ξ_0 , u_p and g_0 are all defined on the support of β .

Lemma 7.1. *The pairing (7.4) is well-defined, i.e., independent of the choice of representative $g_0 \in E_0$ of g , and $\xi_0 \in \ker \varphi_{p+1}^*$ of $[\xi_0]$.*

Proof. To see that this pairing indeed is well-defined, we note first that if $g_0 = \varphi_1 g_1$, then if $p > 1$, since $u_p \varphi_1 = \varphi_p u_{p-1} - \bar{\partial} u_p^1$ by (7.1), and $\xi_0 \varphi_p = 0$, we get that $\xi_0 u_p g_0 = \bar{\partial}(\xi_0 u_p^1 g_1)$. Thus, by Stokes' theorem,

$$\int \xi_0 u_p g_0 \wedge \bar{\partial} \beta = \int \bar{\partial}(\xi_0 u_p^1 g_1) \wedge \bar{\partial} \beta = 0.$$

If $p = 1$, then $u_1 g_1 = \text{Id}_{E_1}$, so

$$\int \xi_0 u_1 g_0 \wedge \bar{\partial} \beta = \int \xi_0 g_1 \bar{\partial} \beta = 0,$$

where the last equality holds by Stokes' theorem, since $\xi_0 g_1$ is holomorphic on U , and β has compact support. Thus, (7.4) is independent of the representative $g_0 \in E_0$.

Similarly, if $\xi_0 = \varphi_p^* \xi_1$, then if $p > 1$, we have that $\varphi_p u_p = \bar{\partial} u_{p-1}$, so

$$\int \xi_0 u_p g_0 \wedge \bar{\partial} \beta = \int \xi_1 \varphi_p u_{p-1} \wedge \bar{\partial} \beta = \int \bar{\partial}(\xi_1 u_{p-1}) \wedge \bar{\partial} \beta = 0,$$

where the last equality follows by Stokes' theorem. If $p = 1$, then $\varphi_1 u_1 = \text{Id}_{E_0}$, so

$$\int \xi_0 u_1 g_0 \bar{\partial} \beta = \int \xi_1 g_0 \bar{\partial} \beta = 0,$$

by Stokes' theorem, since $\xi_1 g_0$ is holomorphic on $\text{supp } \beta$. \square

Proposition 7.2. *The pairing (7.4) is functorial in G .*

By functoriality, we mean the diagram similar to (1.4) is commutative for this pairing. The functoriality then holds for $Z \supseteq (\text{supp } F) \cup (\text{supp } G)$. However, in contrast to the pairing in (1.2) and (1.8), where we have the injective map $H_Z^p(\mathcal{O}) \rightarrow H_W^p(\mathcal{O})$ for $Z \subseteq W$, it is not clear that we would have anything similar for the map $\text{Hom}_{\mathbb{C}}(H_{Z^c}^{n, n-p}, \mathbb{C}) \rightarrow \text{Hom}_{\mathbb{C}}(H_{W^c}^{n, n-p}, \mathbb{C})$.

Proof. We consider $\alpha : F \rightarrow G$, where F and G are finitely generated \mathcal{O} -modules of codimension $\geq p$. If $f \in F$, and $\xi \in \text{Ext}^p(G, \mathcal{O})$, then we want to prove that

$$\langle \alpha(f), \xi \rangle_{Lu} = \langle f, \alpha^* \xi \rangle_{Lu}.$$

We let (K, ψ) and (E, φ) be free resolutions of F and G respectively, and let $a : (K, \psi) \rightarrow (E, \varphi)$ be a morphism of complexes extending $\alpha : F \rightarrow G$ as in Proposition 2.5. We also let π denote both the natural surjections $\pi : K_0 \rightarrow \text{coker } \psi_1 \cong F$ and $\pi : E_0 \rightarrow \text{coker } \varphi_1 \cong G$. If $\pi(f_0) = f$, then $\pi(a_0 f_0) = \alpha f$.

It follows from the proof of [L3, Theorem 3.2], that if one for $\ell = p$ and $\ell = p + 1$, lets M_ℓ be the $\text{Hom}(K_0, E_\ell)$ -valued $(0, \ell - 1)$ -form

$$M_\ell = \sum_{k=1}^{\ell-1} (u^E)_\ell^k a_k (u^F)_k^0,$$

which is smooth outside of $Z := (\text{supp } G) \cup (\text{supp } F)$, then

$$u_p^E a_0 - a_p u_p^F = \bar{\partial} M_p - \varphi_{p+1} M_{p+1}$$

outside of Z . One then obtains that

$$\begin{aligned} (\langle \alpha f, \xi \rangle_{Lu})(\beta) &= \int \xi_0 u_p^E a_0 f_0 \wedge \bar{\partial} \beta = \\ &= \int \xi_0 a_p u_p^F f_0 \wedge \bar{\partial} \beta + \int \xi_0 \bar{\partial} M_p f_0 \wedge \bar{\partial} \beta = (\langle f, \alpha^* \xi \rangle_{Lu})(\beta) \end{aligned}$$

where the term involving $\varphi_{p+1} M_{p+1}$ vanishes, since $\xi_0 \varphi_{p+1} = 0$, and the integral involving $\bar{\partial} M_p$ vanishes by Stokes' theorem, and in the last equality, we used that $\alpha^* \xi = [\xi_0 a_p]$. \square

Corollary 7.3. *The pairing (7.4) descends to a pairing*

$$G/G_{(p+1)} \times \text{Ext}^p(G, \mathcal{O}) \rightarrow \text{Hom}_{\mathbb{C}}(H_{Z^c}^{n, n-p}, \mathbb{C})$$

Proof. Proposition 4.2 implies that the surjection $\alpha : G \mapsto G/G_{(p+1)}$ induces an isomorphism $\text{Ext}^p(G, \mathcal{O}) \cong \text{Ext}^p(G/G_{(p+1)}, \mathcal{O})$, cf., for example the proof of Proposition 5.11. Thus, we can write $\xi \in \text{Ext}^p(G, \mathcal{O})$ as $\alpha^* \xi_1 \in \text{Ext}^p(G/G_{(p+1)}, \mathcal{O})$. Thus, if $g \in G_{(p+1)}$, $\alpha(g) = 0$, so

$$\langle g, \xi \rangle = \langle g, \alpha^* \xi_1 \rangle = \langle \alpha(g), \xi_1 \rangle = 0.$$

\square

We now note the following consequence of the results of Lundqvist.

Proposition 7.4. *Let $G = \mathcal{O}/J$, where J has pure codimension p . Then the pairing (7.4) is non-degenerate in the first argument.*

Proof. This follows directly from (7.2). In the proof in [L1] and [L2], this is formulated for $Z = Z(J)$, but from the proof, it is seen that this proof works just as well for any $Z \supseteq Z(J)$ of pure codimension p .

Alternatively, just as in the proof of Lemma 4.8, we could use the theory of linkage, and functoriality of the pairing which is proven below, to reduce non-degeneracy to the complete intersection case from Passare, (1.20). \square

Thus, combining non-degeneracy in the first argument of (3.1) by Lundqvist, and non-degeneracy in the second argument from Passare, (1.20), and the variant of Lemma 4.4 in this setting, we obtain an independent proof of Theorem 1.6 when the pairing is given by (7.4).

Lemma 7.5. *If G has codimension $\geq p$, then the pairing (1.22), defined as the composition of the pairing (1.9) with (1.21) coincides with the pairing (7.4).*

Proof. In order to prove this, we use the representation (6.9) of the pairing (1.9). Taking the image of this under (1.21), and letting it act on $\beta \in H_{Z^c}^{n,n-p}$,

$$R(\langle g, [\xi_0] \rangle_{AW})(\beta) = \int \xi_0 R_p^E g_0 \wedge \beta.$$

By (7.3), and the fact that $\xi_0 \varphi_{p+1} = 0$, we get that

$$R(\langle g, [\xi_0] \rangle_{AW})(\beta) = \int \xi_0 \bar{\partial} U_p g_0 \wedge \beta,$$

and by Stokes, and the fact that $U_p = u_p$ on $\text{supp } \bar{\partial} \beta$, where u_p is smooth, we get that

$$R(\langle g, [\xi_0] \rangle_{AW})(\beta) = \int \xi_0 u_p g \wedge \bar{\partial} \beta = (\langle g, [\xi_0] \rangle_{Lu})(\beta).$$

□

7.3. Proof of Theorem 1.6. Note that non-degeneracy in Theorem 1.5 gives non-degeneracy in Theorem 1.1, so the results of Lundqvist, as discussed in the previous section give non-degeneracy in the first argument in Theorem 1.1 for $G = \mathcal{O}/J$, where J has pure codimension p . We now show that we can always go the other way around as well.

Lemma 7.6. *The map (1.21) is injective.*

Proof. By representing elements in $H_Z^p(\mathcal{O})$ as Coleff-Herrera currents, see (6.1) and using that such currents have the SEP, it is enough to assume that we are on Z_{reg} , and we then choose coordinates such that locally,

$$Z_{\text{reg}} = \{w_1 = \cdots = w_p = 0\} \subseteq \mathbb{C}_z^{n-p} \times \mathbb{C}_w^p.$$

If we take $T \in H_Z^p(\mathcal{O})$, and take its representative $\mu \in CH_Z$, then by [A1, Lemma 3.6], we can write μ as

$$\mu = \sum_{|\alpha| \leq M} a_\alpha(z) \bar{\partial} \frac{1}{w^\alpha}.$$

We then let $z \in Z_{\text{reg}}$ be fixed, and denote the variables (ζ, w) on \mathbb{C}^n , and define the test-form

$$\beta_z^\alpha := \chi(|w|) w^{\alpha-1} dw_1 \wedge \cdots \wedge dw_p \wedge \bar{\partial} \chi(|\zeta - z|) k_{BM}(\zeta - z),$$

where $\chi(t)$ is a cut-off function which is $\equiv 1$ for t sufficiently close to 0, and which is $\equiv 0$ for t sufficiently large, and k_{BM} is the Bochner-Martinelli kernel in $(n-p)$ variables. Since the part of β_z^α depending on $\zeta - z$ is $\bar{\partial}$ -closed, $\bar{\partial} \beta_z^\alpha$ has support on $\text{supp } \chi'(|w|) \cap \text{supp } \chi'(|\zeta - z|)$, which does not intersect Z_{reg} , so $\beta_z^\alpha \in H_{Z^c}^{n,n-p}$.

If we want to show that the map is injective, we thus assume that

$$\int \mu \wedge \beta_z^\alpha = 0,$$

which by the Bochner-Martinelli formula gives that

$$a_\alpha(z) = 0,$$

and thus, the map is injective on Z_{reg} . \square

Remark 7.7. Just as for the previous pairings, if G is a finitely generated \mathcal{O} -module, one can always find a complete intersection ideal $I \subseteq \text{ann } G_{(p)}$ and a surjective morphism $\alpha : (\mathcal{O}/I)^r \rightarrow G_{(p)}$. By the functoriality, one then gets if $g = \pi(f)$ that

$$\langle g, \xi \rangle(\beta) = \langle f, \alpha^* \xi \rangle(\beta)$$

for $\beta \in H_{Z(I)^c}^{n, n-p}$. Note that this does not work for any $\beta \in H_{Z^c}^{n, n-p}$, but just the ones which are $\bar{\partial}$ -closed outside of $Z(I)$.

For any Z of codimension p , one can always find a complete intersection W of codimension p containing Z , but it is not clear to us whether for $\beta \in H_{Z^c}^{n, n-p}$, one can always find a complete intersection $W \supseteq Z$ of codimension p such that $W \cap \text{supp } \bar{\partial}\beta = \emptyset$, and thus, it is not clear that one can always express the residue (7.4) in terms of the pairing for complete intersections as in Example 6.4. When G is Artinian however, this reduction always work, just as in Example 6.7.

Example 7.8. Using the notation from Example 6.7, and the expression (1.18) for the pairing in the complete intersection case, we obtain the following expression,

$$(7.5) \quad \langle g, \xi \rangle_{Lu}(\beta) = \frac{1}{(2\pi i)^n} \int \xi_0 a_n(h e_1 \wedge \cdots \wedge e_n) \wedge B \wedge \bar{\partial}\beta,$$

where

$$B(z) = \frac{\sum (-1)^{k-1} \overline{z_k^{N_k}} \widehat{dz_k^{N_k}}}{(|z_1^{N_1}|^2 + \cdots + |z_n^{N_n}|^2)^n},$$

where $\widehat{dz_k^{N_k}}$ means that $\overline{dz_k^{N_k}}$ is removed from $\overline{dz_1^{N_1}} \wedge \cdots \wedge \overline{dz_n^{N_n}}$. Alternatively, using the expression (1.17), we obtain the following expression,

$$(7.6) \quad \langle g, \xi \rangle_{Lu}(\beta) = \frac{1}{(2\pi i)^n} \int_{\cap \{|z_i|=\epsilon_i\}} \frac{\xi_0 a_n(h e_1 \wedge \cdots \wedge e_n)}{z_1^{N_1} \cdots z_n^{N_n}} \wedge \beta,$$

for any $(n, 0)$ -form β which is holomorphic near $\{0\}$. This type of explicit expression was used by Lejeune-Jalabert to obtain explicit expressions for the fundamental cycle of Artinian ideals, see [LJ2, p. 239].

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